Individual and Social Deprivation when Choices Matter (In progress: do not cite yet!)

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Abstract

Choices are fundamental in explaining differences in the attainments of individuals. This paper addresses the challenge of measuring individual and social deprivation, when keeping individuals responsible for their choices. I show that intuitive principles of distributive justice force social deprivation to be measured by the sum of specific indices of individual deprivation which: (i) respect the preferences of individuals; (ii) compare individuals based on the set of attainments they are deprived of; and (iii) are continuous and convex. Furthermore, I extend the characterization results to differences in needs and categorical dimensions of deprivation and illustrate the criteria using the Norwegian register data.

Keywords: Multidimensional deprivation; responsibility; fairness; differences in needs.

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1 Introduction

Are poor people responsible for their poverty? When looking at developing countries, most people agree that poor have virtually no way to exit poverty on their own, due to no access to clean water, sanitation, education, etc.\(^1\) When looking at the richest countries, there is no consensus. Surprisingly, a large share of society believes that poverty is often the result of people’s “bad choices.”\(^2\) The debate around the responsibility for poverty is central to policy intervention. If poor are largely responsible for their status, cutting the US Medicaid program can be defended on the ground of incentivizing individuals to exert more effort and find their way out of poverty. If instead poverty is mostly a structural phenomenon for which some people are forced into a status of low income and services, Medicaid can be defended on the ground of insurance or equity considerations.

The debate over responsibility for poverty also reaffirms the pitfalls of standard poverty measures. Is it possible to reconcile the concept of poverty as “being deprived of something” with individuals making choices that affect their status? This paper addresses this challenge. It proposes and characterizes measures of individual and social deprivation that keep individuals accountable for their choices.

The key difficulty for accommodating individuals’ choices is the necessity of multiple dimensions of attainments. Among economists and philosophers, there is a wide consensus that individuals’ deprivation status cannot be established uniquely based on income. People differ also in other dimensions, such as health, access to housing, education, liberties, etc. There is little
\(^1\)When people struggle with survival, “necessity displaces desire” as recently suggested by Allen (2017) (see also Deaton (2016)). In such cases, the lack of good-enough alternatives is the main driver of poverty.
\(^2\)The literature on “attribution for poverty” examines the social distribution of opinions about the determinants of poverty. For the US, people seem to recognize that there are multiple determinants of poverty, but tend to attribute more importance to “internal” causes (laziness, dropping out of school, getting married or having children too early, etc.) rather than “external” causes (discrimination, lack of opportunities, low quality of education, etc.); see Feagin (1975) and Smith and Stone (1989). The limits of this view is confirmed by recent experimental evidence showing that many “internal” causes might be driven by differences in cognitive ability (Mani et al., 2013).
consensus, however, on how to accommodate and combine these dimensions. Clearly, the necessity of reporting and summarizing the achievements of the fight against poverty led to a significant number of proposals (see the recent survey by Alkire et al. (2015)). At the same time, more and more authors express concerns for a number of drawbacks that have emerged (Alkire and Foster (2011b); Ravallion (2011); Thorbecke (2011); Aaberge and Brandolini (2015); Cowell (2015); Duclos and Tiberti (2016)), suggesting that these indices should not be used to decide who deserves social intervention and how to choose policies. I shall discuss these drawbacks in Section 4.

The family of criteria characterized here avoids these drawbacks. Each criterion is defined by an intuitive and transparent set of ethical choices, which I illustrate next.

Identification of the deprived individuals. The first ethical choice consists of defining the “no-deprivation set.” The no-deprivation set is the set of attainments that ensure a (sufficiently) good quality of life to any individual. Any individual with such attainments are non deprived. Moreover, if an individual would refuse to switch to some alternative in the no-deprivation set, she is also considered non deprived. These individuals are the “non-deserving poor.” Note that, independently of having such an option, what matters is that a non-deserving poor would not want to exercise it. The definition of the no-deprivation set is related to and generalizes the standard practice of setting a (multidimensional) poverty line.

Comparisons of individual deprivations. The next ethical choice requires determining a family of “iso-deprivation contours.” These sets can be thought of as production isoquants in firm theory. Their role is to allow comparisons of deprivations across individuals. The idea goes as follows. Consider an iso-deprivation contour $D$. Assume individual $i$ would be willing to switch to any of the attainments in $D$. In contrast, individual $j$ would refuse switching for some of the attainments in $D$. Then, individual $i$ is more deprived than individual $j$. The form of the iso-deprivation contours express the substitutability/complementarity of the different dimensions of deprivation. For instance, it is natural to consider perfectly substitutable meals that provide the same nutritional input. In contrast, one might not
want to set a linear trade-off between health and income: for instance, there might be sufficiently large reductions in health that no amount of money can compensate for and that necessarily make an individual more deprived.

**Priority among deprived individuals.** The last ethical choice pertains the sensitivity to the concentration of deprivations. How much more importance should society place to the deprivation of the most deprived individuals? This choice is standard in the unidimensional poverty literature (see Foster et al. (1984)) and is here captured by an increasing and convex function, named “priority function.”

The main result of the paper is to show that the above ethical choices (and the corresponding indices) are necessary and sufficient to satisfy a set of intuitive and compelling axioms. **Preference responsibility** requires society to respect how individuals would make choices. **Continuity** says that small changes of individuals’ attainments lead to small changes in the level of social deprivation. **Equal-preference transfer** is a multidimensional version of the Pigou-Dalton principle, restricted to individuals with same preferences: it requires society to prioritize the most deprived individuals. An additional axiom is also standard: **separability** tells that the ranking of two alternatives is independent of the attainments of an individual who is unconcerned by the choice.

The central and novel axiom is **deprivation fairness.** First, society ought to take into account the deprivation of each individual. Second and more importantly, society cannot consider the deprivation of some individual more important than that of others. The intuition is simple. Assume individual $i$ achieves a certain level of well-being and is the only one deprived in society. Assume that, no matter which attainments individual $i$ has (ensuring that level of well-being), society always measures a larger social deprivation when some other individual $j$ has larger attainments (and $j$ being the only deprived). Then, the deprivation of $i$ would count relatively less than that of $j$. **Deprivation fairness** prevents this kind of discrimination.

There are situations, however, where the lack of resources of some is socially more relevant than for others. This is the case of differences in needs. Differences in needs emerge due to household size and composition, health
status, etc. For equal income, families with children are typically regarded as more income-deprived than families without children. For equal living space, families with two teenagers are typically regarded as more space-deprived than families with two toddlers. To account for such situations, I build on the concept of basic needs. For each type of household, the basic needs is the set of large-enough attainment that ensure that the household is not deprived. Generalizing deprivation fairness, I use basic needs to construct inter-household comparability of deprivation levels and extend the family of deprivation indices to differences in needs.

Finally, I also extend the results to categorical attainments. This extension ensures applicability of the criterion to attainments, such as discrete levels of health, dicotomous indices of access to education, etc.

In the next section, I illustrate the criteria and place the contribution within the literature; I also use the Norwegian register data to discuss and apply the results. Section 3 presents the framework, the axioms, and the main result. Section 4 extends the results to differences in needs and categorical dimensions of deprivation. Section 5 concludes. All proofs are contained in the appendix.

2 Model, axioms, and characterization

2.1 The model

A society consists of a finite set of individuals $N = \{1, \ldots, n\}$ with $n \geq 3$. Each individual $i \in N$ is characterized by a preference relation $R_i$; $R_i$ is a weak order on the $m$-dimensional Euclidean attainments space $X \equiv \mathbb{R}^m_+$, with $m$ finite. The strict preference and indifference relations induced by $R_i$ are denoted by $P_i$ and $I_i$. Each preference relation $R_i$ can be represented by a strictly increasing and concave numerical function $u_i : X \to \mathbb{R}$ such that $\lim_{|x_i| \to \infty} u_i(x_i) = \infty$. Moreover, for each individual $i$ with preferences $R_i$,

\footnote{$\mathbb{R}^m_+$ denotes the non-negative orthant of $\mathbb{R}^m$.}
there is a different individual with the same preferences.4

A social state \( x_N \equiv (x_1, \ldots, x_n) \) assigns an attainments vector \( x_i \) to each individual \( i \in N \). The set of all possible social states is \( X_N \equiv X^n \). A (social) deprivation ranking, denoted \( \succeq \), is a weak ordering of social states. For each pair of social states \( x, x' \in X_N \), \( x \succeq x' \) means that \( x \) is characterized by at least as much deprivation as \( x' \). The asymmetric and symmetric relations induced by \( \succeq \) are denoted \( \succ \) and \( \sim \). A (social) deprivation index, denoted \( D : X_N \rightarrow \mathbb{R} \), is a numerical representation of the deprivation ranking \( \succeq \); that is, \( x \succeq x' \) holds if and only if \( D(x) \geq D(x') \).

2.2 The axioms

The first axiom says that if each individual finds her attainments vector at \( x \in X_N \) at least as desirable as her attainments vector at \( x' \in X_N \), then the social state \( x \) cannot have more social deprivation than the social state \( x' \). Said differently, a necessary condition for having more deprivation is that at least one individual is worse off.

Preference responsibility: For each pair \( x, x' \in X_N \), \( i R_i x' \) for each \( i \in N \) implies \( x' \succeq x \).

Next, small changes in the social state do not cause large jumps in the level of social deprivation.

Continuity: For each \( x \in X_N \), the set \( \{x' \in X_N \mid x' \succeq x\} \) and the set \( \{x' \in X_N \mid x \succeq x'\} \) are closed.

Next, the deprivation ranking of two social states is independent of the attainments vector of an individual who is unconcerned by these alternatives.5

4This richness requirement can be replaced by a duplication invariance axiom: the criterion should rank consistently any two alternatives for the original society and the duplicated alternatives for the duplicated society.

5I adopt the following notation: for each \( x \in X_N \), each \( i \in N \), and each \( a_i \in X \), \( (a_i, x_{-i}) \in X_N \) denotes the social state that assigns \( a_i \) to \( i \) and \( x_j \) to each \( j \in N \setminus \{i\} \).
Separability: For each pair \( x_N, x'_N \in X_N \), if there is \( i \in N \) such that \( x_i = x'_i \equiv a_i \), then for each \( b_i \in X \),

\[
(a_i, x_{-i}) \succeq (a_i, x'_{-i}) \iff (b_i, x_{-i}) \succeq (b_i, x'_{-i}) .
\]

The next axiom reinterprets the Pigou-Dalton transfer principle. Dalton (1920) suggested that a progressive transfer from a richer to a poorer individual (provided the richer/poorer relation is not reversed) leads to a more desirable distribution of income. Here, I impose that such transfer of attainments among equal-preference individuals weakly reduces social deprivation.

Equal-Preference Transfer: For each pair \( j, k \in N \) such that \( R_j = R_k \equiv R_0 \), if there exist a pair \( x_N, x'_N \in X_N \) and \( \alpha \geq 0 \) such that:

(i) \( x_j - \alpha (x_j - x_k) = x'_j \) \( R_0 x'_k = x_k + \alpha (x_j - x_k) \);

(ii) for each \( i \in N \setminus \{j, k\} \), \( x_i = x'_i \);

then \( x_N \succeq x'_N \).

The last axiom introduces deprivation and prevents a certain form of discrimination: the deprivation of some individual cannot be considered more important than that of others. First, there exists a subset of (sufficiently large) attainments vectors \( C \) such that whether an individual is assigned such attainments or more does not affect social deprivation. Second, consider a social state where everyone is assigned such a sufficiently large attainments vector, i.e. \( x^*_N \in C^n \). Let now an individual \( i \in N \) be given a different attainments vector \( \tilde{x}_i \in X \) so that social deprivation is larger, that is \( (\tilde{x}_i, x^*_i) \succ x^*_N \). Then, \( i \) is discriminated against if, for each attainments vector \( x_i \in X \) that she finds equally desirable (i.e. such that \( x_i I_i \tilde{x}_i \)), there is less social deprivation at the social state \( (x_i, x^*_{-i}) \) than at any social state \( (x_j, x^*_{-j}) \in X_N \) which assigns a larger attainments vector \( x_j > x_i \) to some other individual \( j \in N \setminus \{i\} \).

6Vector inequalities are denoted \( \succeq, >, \text{ and } \gg \).
lead to a lower level of social deprivation. Such discrimination is prevented here.

**Deprivation fairness:** There exists a non-empty and closed set \( C \subset X \) such that:

\[(i) \quad \text{for each } x_i^* \in C, \text{ each } i \in N, \text{ and each } x'_i \in X \text{ with } x'_i \geq x_i^*, \ (0, x_{-i}) \succ (x'_i, x_{-i}^*) ; \]

\[(ii) \quad \text{for each } x_i^* \in C, \text{ each } i \in N, \text{ and each } \tilde{x}_i \in X \text{ with } (\tilde{x}_i, x_{-i}) \succ (x_i^*, \tilde{x}_i^*) , \text{ there exists } x_i \in X \text{ with } x_i \succ \tilde{x}_i \text{ such that for each } j \in N \setminus \{i\} \text{ and each } x_j \in X, x_i < x_j \text{ implies } (x_i, x_{-i}^*) \succ (x_j, x_{-j}^*). \]

### 2.3 Individual and social deprivation

I next define the family of criteria. Let the **no-deprivation set** is a non-empty, closed, and convex subset \( NDS \subset X \) such that, if \( x \in NDS \) and \( x' \geq x \), then \( x' \in X \).

Let \( \succcurlyeq \) be a weak order on \( X \) that is continuous, strictly antimonotonic on \( X \setminus NDS \) (i.e. for each \( x, x' \in X \setminus NDS, x \geq x' \) implies that \( x' \succ x \)), constant on \( NDS \) (i.e. for each \( x, x' \in NDS, x \sim x' \)), and convex (for each \( x \in X, \{x' \in X | x \succ x' \} \) is convex). The weak order \( \succcurlyeq \) identifies a sequence of **iso-deprivation contours**, that is, sets of attainments equally ranked by \( \succcurlyeq \). Let \( \lambda : X \to [0, 1] \) be a representation of \( \succcurlyeq \) such that, without loss of generality, \( \lambda (x) = 0 \) for \( x \in NDS \) and \( \lambda (0) = 1 \). Let \( \lambda_i : X \to [0, 1] \) be \( i \)'s **individual deprivation function** for \( \succcurlyeq \), defined by setting, for each \( x_i \in X, \lambda_i (x_i) = \min \{\lambda (x) | x I_i x_i\}. \)

Let the **normalized least joint convex function** \( \phi \) be a real-valued, continuous, and increasing function such that: \( (i) \ \phi \circ \lambda_i \) is convex for each \( i \in N; \ (ii) \ \text{for } \phi \text{ is least convex among the functions satisfying } (i); \) and, without loss of generality, \( (iii) \ \phi (0) = 0 \) and

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7The existence of a least desirable attainments vector for each iso-deprivation contour is ensured by continuity of \( \succcurlyeq \) and the preferences assumption that, for some numerical representation \( u_i, \lim_{|x_i| \to \infty} u_i (x_i) = \infty. \)
The function \( \phi \) provides cardinal meaning to the interpersonally comparable functions \( \lambda_i \).\(^8\) Let \( f \) be a real-valued and convex function, named priority function.

Each member of the family of deprivation indices characterized here can be written as follows:

\[
D(x_N) = \sum_{i \in N} f \circ \phi \circ \lambda_i(x_i).
\] (1)

To repiligate, the ingredients of (1) are the following:

1. the no-deprivation set \( NDS \) determines that \( \lambda_i(x_i) = 0 \) if \( i \)'s attainments \( x_i \) are large enough \( (x_i \in NDS) \);
2. the iso-deprivation contours determine the extent of deprivation of each individual in a way that is interpersonally comparable: \( i \) is at least as deprived as \( j \) iff \( \lambda_i(x_i) \geq \lambda_j(x_j) \);
3. the priority function \( f \) defines the priorities attributed to individuals based on their cardinalized deprivation indices \( \phi \circ \lambda_i \).

The main result establishes the equivalence between deprivation rankings satisfying the introduced axioms and the social deprivation index in (1).

**Theorem 1.** A deprivation ranking satisfies preference responsibility, continuity, separability, deprivation fairness, and equal-preference transfer if and only if it can be represented by a deprivation index as (1).

\(^8\)A function \( \phi \) is least convex if any convex function \( \psi \) can be written by composing \( \phi \) with a convex function \( f \), i.e. \( \psi = f \circ \phi \). It is unique up to an increasing affine transformation. The normalization of Condition (iii) ensures its uniqueness. See Debreu (1976), Kannai (1977), and Piacquadio (2017).

\(^9\)The possibility of dropping \( \phi \) from the formula (7) in the above example is due to the homotheticity of preferences and of the iso-deprivation contours. Then, the representation of preferences that is homogeneous of degree 1 is also a least concave function. Thus, \( -\lambda_i \) is least convex. As \( \phi \) would be an increasing linear transformation, it can be omitted.
3 Extensions

3.1 Differences in needs

As Atkinson writes (1987, p.753): “it should be noted that...families have been assumed to be identical in their needs and the poverty line has been taken as the same for all. In practice, the poverty line is different for families of different size and differing in other respects. There is therefore scope for disagreement not just about the level of the poverty line but also about its structure.”

To tackle differences in needs, I introduce the following changes. First, I weaken the axioms which derive their normative appeal from the implicit assumption of equal needs. Second, I introduce information about differences in needs and, correspondingly, axioms that allow incorporating such information in the deprivation ranking.

The first change is straightforward. Deprivation fairness and equal-preference transfer should be restricted to individuals with the same needs. By doing so, the axioms would lead to a need-specific no-deprivation set and to need-specific iso-deprivation contours. Thus, whenever two individuals are characterized by the same needs, their deprivations are compared through the need-specific iso-deprivation contours. This leaves open the question of how to compare and aggregate deprivations across individuals with different needs.

The second change is more challenging. The standard answer is to base such comparisons on equivalence scales (see Lewbel (1989) and Ebert and Moyes (2003)). Equivalence scales, however, are constructed by assuming interpersonal comparability of utilities (or equivalently deprivation levels): needs are then implicitly defined by the different quantity of commodities needed to achieve the same level of utility. Here, no information is assumed about how to make interpersonal comparisons of utilities, which, instead, emerge endogenously from the axioms. Abiding by this approach, I suggest that the only additional information available is the set of attainments vectors that are considered sufficient to cover each individual’s “basic needs,” a
multidimensional version of Atkinson’s family-specific poverty lines.

Let \( \Theta \) denote a finite set of types. The population is correspondingly partitioned, that is, there are a finite number of non-empty and disjoint set of individuals \( N^\theta \) such that \( N = \bigcup_{\theta \in \Theta} N^\theta \). For each \( \theta \in \Theta \), \( N^\theta \) consists of \( n^\theta \) individuals satisfying the assumptions of Section 2. For each \( \theta \in \Theta \) and each \( i \in N^\theta \), let the basic needs of \( i \) be a closed and non-empty set of attainments \( B^\theta \subset X \). Let \( B = (B^\theta)_{\theta \in \Theta} \) define the basic needs of each type of individuals. For each \( \theta \in \Theta \) and each \( \alpha > 0 \), let \( \alpha B^\theta \equiv \{ x \in X \mid x = \alpha b \text{ with } b \in B^\theta \} \). With a slight abuse of notation, let \( \alpha B^\theta P_i x_i \) mean that \( i \) considers all attainments in \( \alpha B^\theta \) more desirable than \( x_i \) and let \( \alpha B^\theta \neg P_i x_i \) denote its negation, that is, there is at least an alternative in \( \alpha B^\theta \) that \( i \) finds at least as desirable as \( x_i \).

I can now state formally a version of deprivation fairness that accounts for differences in needs. The first condition is unchanged. The second condition is only imposed on individuals belonging to the same type. The third condition is new and accounts for individuals with different needs. Let individual \( i \in N^\theta \) find her attainments vector insufficient to cover her basic needs \( B^\theta \), that is \( B^\theta P_i x_i \). In contrast, this is not true for individual \( j \in N^{\theta'} \), that is \( B^{\theta'} \neg P_j x_i \). Then, \( i \) should be considered at least as deprived as \( j \). Importantly, this requirement is also imposed for proportional expansions and contractions of the set of basic needs, that is when for some \( \alpha > 0 \), \( \alpha B^\theta P_i x_i \) and \( \alpha B^{\theta'} \neg P_j x_j \), meaning that the relation between needs of individuals is preserved under rescaling of attainments.\(^{10}\)

**Needs-adjusted deprivation fairness:** For each \( \theta \in \Theta \), there exists a non-empty and closed set \( C^\theta \subset X \) satisfying the following conditions. Let \( C^\Theta_X \equiv \{ x_N \in X^n \mid x_i \in C^\theta \text{ for each } \theta \in \Theta, i \in N^\theta \} \).

\(^{10}\)This rescaling condition is demanding. It is not obvious that proportional changes of the set of basic needs preserve interpersonal comparability in terms of deprivation of resources. Assume two different families have basic needs of, respectively, 1000$\$/month and 800$\$/month. Then, in a one-dimensional setting, this proportionality assumption also imposes that if these families earn, respectively, 500$\$/month and 400$\$/month, they are equally poor. In a multidimensional setting, preference diversity may complicate this relationship, but the idea remains controversial. Yet, whenever one can assess that proportionality is not desirable, more information is available on how deprivation changes with size. This information can then be used to avoid proportionality.
Then:

(i) for each $x_N^* \in C_N^\Theta$, each $\theta \in \Theta$, each $i \in N^\theta$, and each $x'_i \in X$ with $x'_i \geq x^*_i$, $(0, x^*_i) \triangleright x_N^* \succeq (x'_i, x^*_i)$;

(ii) for each $x_N^* \in C_N^\Theta$, each $\theta \in \Theta$, each $i \in N^\theta$, and each $\bar{x}_i \in X$ with $(\bar{x}_i, x^*_i) \triangleright x_N^*$, there exists $x_i \in X$ with $x_i I_i \bar{x}_i$ such that for each $j \in N_0 \setminus \{i\}$ and each $x_j \in X$, $x_i < x_j$ implies $(x_i, x^*_i) \triangleright (x_j, x^*_j)$;

(iii) for each $x_N^* \in C_N^\Theta$, each $\alpha > 0$, each pair $\theta, \theta' \in \Theta$, each $i \in N^\theta$, each $j \in N^{\theta'}$, and each pair $x_i, x_j \in X$ with $\alpha B^\theta P_{i} x_i$ and $\alpha B^{\theta'} \neg P_{j} x_j$, then $(x_i, x^*_i) \succeq (x_j, x^*_j)$.

Condition (iii) of needs-adjusted deprivation fairness introduces differences in needs for the measurement of individual and social deprivation. Combined with the other axioms, it forces the no-deprivation sets (one for each type) and the iso-deprivation contours to reflect the basic needs of each individual. In fact, for each type, the iso-deprivation contours are all homothetic and consist of the convex closure of the set of basic needs. What is left undefined is the size of the type-specific no-deprivation sets, which can be any common proportional expansion (for instance, when looking at relative poverty) or contraction (for instance, when looking at absolute poverty) of the basic needs.

Before stating this result, I present the multidimensional transfer principle. As anticipated, the only difference with equal-preference transfer is that transfers are now restricted to individuals with same preferences and same needs.

Transfer among equals: For each $\theta \in \Theta$ and each pair $j, k \in B^\theta$ with $R_j = R_k \equiv R_0$, if there exist a pair $x_N, x'_N \in X_N$ and $\alpha \geq 0$ such that:

(i) $x_j - \alpha (x_j - x_k) = x'_j R_0 x'_k = x_k + \alpha (x_j - x_k)$;

(ii) for each $i \in N \setminus \{j, k\}$, $x_i = x'_i$;
Next, I define the deprivation indices that accommodate differences in needs. The type-specific no-deprivation set is defined by \( \bar{\alpha}_B^\theta \) for each \( \theta \in \Theta \): it is proportional to the basic needs of each type of households. Let the index of deprivation for individual \( i \in N \) of type \( \theta \in \Theta \) be:

\[
\lambda_i^\theta (x_i) = \max \left\{ 0, 1 - \frac{\alpha}{\bar{\alpha}} \left| x_i I_i x \right| \text{ for some } x \in \alpha B^\theta \right\}.
\]

Let \( \phi \) be the normalized least joint convex transformation associated to these deprivation indices. Formally, \( \phi \) is a real-valued, continuous, and increasing function such that: (i) \( \phi \circ \lambda_i^\theta \) is convex for each \( \theta \in \Theta \) and each \( i \in N^\theta \); (ii) \( \phi \) is least convex among the functions satisfying (i); and, without loss of generality, (iii) \( \phi (0) = 0 \) and \( \phi (1) = 1 \). Finally, let \( f \) be the priority function, i.e., a continuous, increasing, and convex function. Then, the deprivation index \( D \) can be defined by setting for each \( x_N \in X_N \):

\[
D (x_N) = \sum_{\theta \in \Theta} \sum_{i \in N^\theta} f \circ \phi \circ \lambda_i^\theta (x_i). \tag{2}
\]

The following result states that preference responsibility, continuity, separability, and the modified versions of deprivation fairness and equal-preference transfer characterize the deprivation index (2).

**Theorem 2.** A deprivation ranking satisfies preference responsibility, continuity, separability, and the modified versions of deprivation fairness and equal-preference transfer if and only if it can be represented by a deprivation index as (2).

### 3.2 Categorical attainments

In the measurement of individual and social deprivation, it is often the case that attainments are categorical variables (see, among others, Alkire and Foster (2011a); Bossert et al. (2013); and Decancq et al. (2014)). I briefly address next how to extend the results when some dimensions are categorical.

Let \( S \) be the set of all possible combinations of categorical attainments,
where the cardinality of $S$ is finite. Define the extended attainments space as $X^+ \equiv X \times S$. Preferences $R_i$ of each individual $i \in N$ are now defined on $X^+$ and are such that, for each categorical attainments $s \in S$, the weak order of attainments in $X$ satisfies the assumptions of Section 2. Moreover, for each $x \in X$, each pair $s, s' \in S$, there exists $x' \in X$ such that $(x, s) \sim_i (x', s')$. 

A social state specifies an attainments vector $x^+_i \equiv (x_i, s_i) \in X^+$ for each individual $i \in N$. A deprivation ranking $\succeq$ is a weak ordering of social states.

A deprivation index represents such ranking by a function $D : X^+_N \to \mathbb{R}$. I suggest the axioms be changed as follows.

- **Preference responsibility**$^+$ and **separability**$^+$ are imposed on the extended attainments space $X^+$. These axioms are independent of the nature of the dimensions and do not pose any difficulties.\(^{11}\)

- **Continuity**$^+$ and **equal-preference transfer**$^+$ are instead imposed on the space of continuous dimensions $X$ (for each given vector of categorical attainments). Said differently, for each $s_N \in S^N$, the projection of the deprivation ranking on $X^N$ satisfies the continuity and equal-preference transfer axioms introduced previously.\(^{12}\)

- **Deprivation fairness**$^+$ is instead imposed for a reference vector of categorical attainments, equal across individuals. That is, there exists a vector of categorical attainments $s_N^* \in S_N$ with $s_i^* = s^*$ for each $i \in N$ that is kept fixed when imposing the previous version of deprivation fairness. As an example, if health is a categorical variable, $s^*$ might be chosen as the attainment level corresponding to perfect health (as in Fleurbaey and Schokkaert (2009)).\(^{13}\)

\(^{11}\)As an example, *preference responsibility* now demands that for each pair $x^+_N, x^+_N \in X^+_N$, $x^+_N \preceq x^+_N$ for each $i \in N$ implies $\bar{x}^+_N \succeq \bar{x}^+_N$.

\(^{12}\)As an example, *continuity* now demands that for each $x^+_N \equiv (x_N, s_N) \in X^+_N$, the sets $\{\bar{x}^+_N \equiv (\bar{x}_N, \bar{s}_N) \in X_N \mid \bar{x}_N \succeq x^*_N \text{ with } s_N = \bar{s}_N\}$ and $\{\bar{x}^+_N \equiv (\bar{x}_N, \bar{s}_N) \in X_N \mid \bar{x}_N \succeq \bar{x}^*_N \text{ with } s_N = \bar{s}_N\}$ are closed.

\(^{13}\)Formally, *deprivation fairness* now demands that there exists a non-empty and closed set $C \subset X$ and $s^*_N \in S_N$ with $s_i^* = s^*$ for each $i \in N$ such that: (i) for each $x^*_N \in C^N$, each $i \in N$, and each $x_i \in X$ with $x_i^* \succeq x^*_i$; $(0, s^*)$, $(x^*_i, s^*_i) \triangleright (x^*_N, s^*_N)$; and (ii) for each $x^*_N \in C^N$, each $i \in N$, and each
Then, the main result goes through. More precisely, the deprivation ranking satisfies the modified axioms if and only if it can be represented by a deprivation index
\[ D^+ (x^+_N) = \sum_{i \in N} f \circ \phi \circ \tilde{\lambda}_i (x^+_i), \] 
(3)
where:

- for a given \( s^* \in S, \tilde{\lambda}_i (x^+_i) \) is defined by setting for each \( x^+_i \equiv (x_i, s_i) \in X^+ \):
  \[ \tilde{\lambda}_i (x^+_i) = \min \{ \lambda (x) \mid (x, s^*) I_i x^+_i \} \]; and

- \( \phi \) is the normalized least joint convex function on \( X \), that is a real-valued, continuous, and increasing function such that: (i) \( \phi \circ \tilde{\lambda}_i \) is convex on \( X \) for each \( i \in N \); (ii) \( \phi \) is least convex among the functions satisfying (i); and, without loss of generality, (iii) \( \phi (0) = 0 \) and \( \phi (1) = 1 \).

- \( f \) is the priority function, that is a continuous, increasing, and convex function.

**Theorem 3.** A deprivation ranking on \( X^+_N \) satisfies preference responsibility\(^+\), continuity\(^+\), separability\(^+\), deprivation fairness\(^+\), and equal-preference transfer\(^+\) if and only if it can be represented by a deprivation index as (3).

To clarify, the no-deprivation set and the iso-deprivation contours continue to be defined on the subspace of continuous dimensions \( X \). The reason is that only \( X \) has the properties required to define interpersonal comparability and cardinality of deprivations based on individuals’ attainments vectors. Yet, this doesn’t prevent the family of deprivation indices to keep individuals responsible for their choices. In fact, the deprivation indices allow comparisons over the entire space of categorical attainments by “Pareto indifference,” which demands that whenever all individuals are indifferent between two social states, social deprivation is unchanged.

\( \bar{x}_i \in X \) with \( ((\bar{x}_i, s^*), (x^*_{-i}, s^*_{-i})) \triangleright (x^*_N, s^*_N) \), there exists \( x_i \in X \) with \( (x_i, s^*_i) I_i (\bar{x}_i, s^*_i) \) such that for each \( j \in N \setminus \{i\} \) and each \( x_j \in X, x_i < x_j \) implies \( ((x_i, s^*), (x^*_{-i}, s^*_{-i})) \triangleright ((x_j, s^*), (x^*_{-j}, s^*_{-j})) \).
4 Related literature and empirical illustration

4.1 Income poverty

Poverty is most commonly measured in terms of after-tax income. This is a one-dimensional setting \(m = 1\). An income distribution \(y_N \equiv (y_1, ..., y_n) \in \mathbb{R}_+^n\) assigns a certain level of income \(y_i \geq 0\) to each individual \(i \in N \equiv \{1, ..., n\}\).

How to measure poverty? Following the literature, the first step is to set a poverty line \(z_y > 0\): individuals with income larger than \(z_y\) are non-deprived; individuals with income smaller or equal to \(z_y\) are income-deprived. The second step is to define each individuals’ income gap as the difference (if any) between the poverty line and the assigned income; formally, \(d(y_i) = z_y - y_i\) if \(z_y \geq y_i\) and \(d(y_i) = 0\) otherwise.

The last step is to aggregate each individuals’ income gap. An intuitive way to aggregate income deprivation is the mean income gap:

\[
D(y_N) = \frac{1}{n} \sum_i d(y_i).
\] (4)

A possible drawback of (4) is its insensitivity to the concentration of income gap among individuals. Said differently, poverty is unchanged whether a certain income gap is held by few individuals or is shared among many. To avoid this drawback, the simple average can be replaced by the generalized mean income gap, defined by:

\[
D_f(y_N) = \frac{1}{n} \sum_i f(d(y_i)),
\] (5)

where \(f\) is an increasing and convex real-valued function. When \(f\) is linear, society is unconcerned with the distribution of income gaps and (5) is ordinally equivalent to (4). As \(f\) becomes more and more convex, society places more and more weight on the individuals with largest income gaps. At the limit for \(f\) infinitely convex, poverty is defined by the income gap of the most deprived individual in society. Interestingly, in the one-dimensional setting, the criterion characterized in this paper is equivalent to (5).
A related family of poverty indices is that by Foster et al. (1984):

\[ D_{FGT} (y_N; \theta) = \frac{1}{n} \sum \left( \frac{d(y_i)}{z_y} \right)^\theta. \]  

(6)

Two remarks are in order. First, the Foster-Greer-Thorbecke includes as a special case the head-count index (when \( \theta = 0 \)), which measures social deprivation by the proportion of individuals that are income deprived. It accommodates strict aversion to inequality in deprivations only if \( \theta > 1 \). Second, the distinguishing aspect of (6) is “scale invariance” (see Ebert and Moyes, 2002). Scale invariance requires the deprivation index to be unchanged when rescaling the income distribution (together with the poverty line).\(^{14}\)

When evaluating income distributions, the deprivation measures characterized here are equivalent to the generalized mean income gap (5). Intuitively, with one dimension, monotonicity of preferences eliminates differences in choices. The no-deprivation set is then the set of all incomes larger than a threshold and non-poor individuals are those having sufficiently large incomes. The income gap is the only way of making interpersonal comparisons of deprivation that is consistent with deprivation fairness (in one dimension, it is weaker than anonymity). Finally, the linearity of the income gap ensures that this is a least convex representation of preferences, requiring social deprivation to be the sum of its convex transformation.

**Empirical illustration.**

I use the 2016 Norwegian register data. I focus on the universe of Norwegian single men that do not have children and are aged between 22 and 61.\(^{15}\) My database consists of 90462 individuals.

Following practice, set the poverty line at 60 percent of the median after-

\(^{14}\)Despite its popularity, scale invariance is not compelling in a multidimensional setting and is not imposed here. The problem is that, with several commodities, individual \( i \) may prefer an attainments vector \( x_i \) to a different one \( x'_i \) and, at the same time, prefer the rescaled bundle \( \alpha x'_i \) to \( \alpha x_i \) for some \( \alpha > 0 \). This preference inversions make it impossible for any scale invariant criterion to respect individuals’ preferences.

\(^{15}\)The focus on a specific category of individuals significantly reduces the scope for accounting for differences in needs. I leave to future research the empirical assessment of deprivation in Norway using all types of households.
tax income. In the sample, the median after-tax income is $y_m = 0.333$ million NOK and the poverty line is $z_y = .2$ (this corresponds to about 24,000 USD). The mean income gap is $D(y_N) = .0011$; if the total income gap was distributed equally in the population, each individual would have an income gap of 1,100 NOK (about 130 USD).

The head-count index, corresponding to (6) with $\theta = 0$, is $D_{FGT}(y_N; 0) = .0297$. This tells that 2692 of the 90463 individuals are income poor.

Continuing with the family of poverty measures (6), when $\theta = 1$, the relative mean income gap (equivalent to the mean income gap up to a multiplicative constant) is $D_{FGT}(y_N; 1) = z_y^{-1}D(y_N) = .0055$.

Introducing priority for the poorest individuals requires $\theta > 1$. Let $\theta = 2$.

This means that there is roughly the same level of income poverty whether: (a) individual $i$ has no income while $j$ is not poor or (b) whether $i$ and $j$’s incomes are both 30 percent of the poverty line. In the data, $D_{FGT}(y_N; 2) = .0012$.

### 4.2 Multi-dimensional poverty in the literature

When two or more dimensions are considered, there is no consensus on how to measure poverty. An index of multidimensional poverty needs to aggregate the attainments of each individual (an $m \times n$ dimensioned social state) into a real value. It follows that a crucial ethical choice is the order of aggregation: one can aggregate over individuals first or over dimensions first.

The first aggregation method—over individuals first—closely builds on the one-dimensional criteria. Poverty is first assessed for each dimension separately: in order, setting a dimension specific poverty line, computing the dimension-specific attainment’s gap of each individual, and aggregating these in a measure of dimension-specific poverty such as (4) or (6). In a second stage, these dimension-specific poverty levels are aggregated into a composite or “mashup” index.\(^\text{16}\) A well-known example is the Human Development

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\(^{16}\)Some authors argue that aggregation across dimensions is *ad hoc* and unnecessary. They instead suggest reporting the entire vector of dimension-specific deprivations (Hicks and Streeten, 1979). Unfortunately, this “dashboard” approach is often unable to provide a clearcut comparison of social states and, in particular when many dimensions are
Importantly, Mashup indices disregard the joint distribution of deprivations across individuals, which means that they are unconcerned whether the dimension-specific deprivations are borne by few individuals only or spread out among many individuals.

To avoid this drawback, Alkire and Foster (2011a) propose the adoption of a “dual cutoff method.” The first cutoff is standard: a vector of dimension-specific poverty lines defines which dimensions (if any) an individual is deprived off and determines her dimension-specific attainment’s gaps. The second cutoff establishes whether the dimension-specific deprivations are sufficient to indentify an individual as (overall) poor. For example, the “intersection approach” tells that an individual is poor if she is deprived in all dimensions. Then, multidimensional poverty is the sum of the (possibly weighted and transformed) dimension-specific attainments’ gaps that each poor individual experiences. With this proposal, the joint distribution of deprivations now matters. Yet, this is limited to the identification of the poor individuals: the distribution of dimension-specific attainments’ gaps among poor individuals is not accounted for.

The second aggregation method—over dimensions first—is closely related to the “social welfare approach” to multidimensional poverty (Atkinson, 2003). Individuals’ attainment gaps are first evaluated through a “utility-like” function and then aggregated across individuals.

A first alternative is to adopt individuals’ cardinal and interpersonally comparable utility functions to evaluate their deprivation (Kingdon and Knight, 2006). The focus is on individuals’ happiness or subjective well-being. The advantage of this approach is that it respects the choices of individuals (as long as individuals choose what gives them largest happiness). Yet, economists and philosophers have convincingly contended that poverty is about the lack of access to resources or attainments, rather than happiness (Sen, 1979).

A second alternative by Bourguignon and Chakravarty (2003) is to let
society choose the utility-like function to evaluate the deprivations of each individual and then aggregate them additively across individuals (see also Duclos et al., 2006). As clarified by Atkinson (2003), this function reflects the preferences of society and incorporates three ethical choices, similar to the ones identified here. First, it requires setting a multidimensional poverty line. The dimension-specific deprivations are \( d_y(y_i) \equiv \max[0, z_y - y_i] \) and \( d_l(l_i) \equiv \max[0, z_l - l_i] \). Second, it requires setting “iso-poverty contours.” Society chooses these level curves to specify which combinations of dimension-specific deprivation are associated the same level of individual deprivation. The form of such curves accommodates different degrees of substitutability and complementarity between dimensions. It can be described by a complete ordering of attainments. Fig.1 illustrates this construction for a two-dimensional setting with income and leisure. For example, individual \( i \) with attainments vector \( x_i \) is on a lower iso-poverty contour than individual \( j \) with attainments vector \( x_j \). Third, society selects a common utility-like function to represent the iso-poverty contours. This choice defines, through its convexity, how quickly the level of individual deprivation changes when climbing the contours.

A first limitation of this approach is to be insensitive to some changes in attainments of poor individuals. Increasing the income of individual \( i \) (from \( x_i \)) doesn’t change her level of poverty, no matter how large her income.

Another limitation is to disregard individuals’ choices. To illustrate, let \( j \)’s attainments vector be above the poverty line, i.e. \( x'_j = (1 + \varepsilon) z \) with \( \varepsilon > 0 \). A policy offers individual \( j \) the possibility to switch to the attainments vector \( x_j \), which \( j \) prefers to \( x'_j \). While \( j \) would exercise her opportunity to be better off, \( x_j \) might be below the poverty line (even if \( x'_j \) was not). In fact, all indices presented above—except the one based on happiness—would allow for poverty to increase when a non-poor individual switches to a better alternative. Does this mean that respecting preferences requires comparing individuals’ deprivations by their happiness levels? As recently clarified by Fleurbaey and Maniquet (2011) and Piacquadio (2017) for the measurement of social welfare, the answer is negative. The trick is to measure deprivation with respect to utility-like functions that are different across individuals.
and represent individuals’ preferences. The contribution of this paper is to characterize how society ought to choose these functions based on fairness principles.

A recent preference-sensitive proposal related to the present one is by Decancq et al. (2014) (see also Dimri and Maniquet (2017)). Two main differences emerge here. First, their poverty line is similar to that introduced by Bourguignon and Chakravarty (2003) and does not allow for complementarity/substitutability across dimensions. Second, society may promote regressive redistributions of attainments, i.e. from more deprived to less deprived individuals (even if they have the same preferences). Their criterion emerges as a special case of the family of indices characterized here when the preferences of individuals are homothetic and when society considers the attainments as perfect complements.

4.3 Empirical application of the present proposal

Three fundamental ethical choices emerge from the characterization result.
Identification of the deprived. An individual is deprived if any bundle from the no-deprivation set would make her better-off. The no-deprivation set is a subset of the attainments space. It includes all attainments that are sufficiently large to ensure that individuals with any such attainments are not deprived.

In one dimension, say income, this is equivalent to setting a poverty line. In the multidimensional setting, it includes as a special case the multidimensional poverty line $z$ (as in Bourguignon and Chakravarty). Here, the no-deprivation set is more permissive: one can flexibly trade-off income and leisure and set any level of complementarity or substitutability between the different dimensions of deprivation. For instance, a sufficiently large income may be considered sufficient to compensate an individual for longer working hours and might ensure that an individual is not deprived.

For the sake of simplicity, let the no-deprivation set $NDS$ be identified by the set of attainments vectors $(y, l) \in X$ such that:

$$f(y, l) \equiv [\beta y^\gamma + (1 - \beta) l^\gamma]^{\frac{1}{\gamma}} \geq \nu.$$  

The parameters $\beta, \gamma, \nu \in \mathbb{R}$ are uniquely defined by the following ethical choices:\footnote{These choices are clearly arbitrary, but illustrate how to apply the present results.}

- when working as much as the median labor supply, any income larger than (or equal to) 60 percent of median income ensures that an individual is not deprived;

- when working 50 percent less than the median labor supply, any income larger than (or equal to) 40 percent of median income ensures that an individual is not deprived;

- when working 33 percent more than the median labor supply, any income larger than (or equal to) the median income ensures that an individual is not deprived.
Together, these conditions imply that \( \beta = 0.177, \gamma = -3, \nu = 0.33 \) and uniquely identify the no-deprivation set.

**Comparison of individual deprivations.** Assume two individuals \( i \) and \( j \) are both deprived. Who is most deprived? The answer comes from the **iso-deprivation contours**. Iso-deprivation contours are level curves similar to the “iso-poverty contours” in Bourguignon and Chakravarty (2003). The iso-deprivation contour corresponding to the lowest level of deprivation corresponds to the no-deprivation set (without loss of generality, this level of deprivation is normalized to 0). The smaller the attainments vector, the closer to the origin is the corresponding iso-poverty contour and the larger is the associated level of deprivation. The iso-deprivation contour corresponding to the highest level of deprivation is the origin (without loss of generality, this is normalized to 1).

Importantly, iso-deprivation contours cannot be directly used to assess individuals’ deprivations, as doing so would not respect individuals’ choices. Consider two individuals, \( i \) and \( j \), with attainments \( x_i, x_j \in X \). Individual \( i \) is considered more deprived than individual \( j \) if there exist an iso-deprivation contour such that \( i \) would be better off with any attainments vector from this contour, while \( j \) would not. Fig. 2 illustrates the construction of the no-deprivation set and the iso-deprivation contours. Individual \( i \) with preferences \( R_i \) is not income-poor. Yet, she is considered deprived since she would be better-off with any attainments vector from the no-deprivation set. Individual \( j \) with preferences \( R_j \) is income-poor. Yet, she is not considered deprived since she would not want to switch from \( x_j \) to \( x_M \) (or other attainments vectors from the no-deprivation set).

Here, I assume homothetic iso-deprivation contours, which are thus directly generated by the no-deprivation set.

Then, the iso-deprivation contour of level \( \lambda \in [0,1] \), denoted \( IDC(\lambda) \), consists of all the attainments vectors \((y, l) \in X\) such that:

\[
\begin{align*}
\left[\beta y^\gamma + (1 - \beta) l^\gamma\right]^\frac{1}{\gamma} &= (1 - \lambda) \nu & \text{if } \lambda > 0, \\
\left[\beta y^\gamma + (1 - \beta) l^\gamma\right]^\frac{1}{\gamma} &\geq \nu & \text{if } \lambda = 0.
\end{align*}
\]
Figure 2: The no-deprivation set and the iso-deprivation contours.

These choices allow to make interpersonal comparisons based on the individual-specific deprivation function $\lambda_i : X \to \mathbb{R}$, be defined as follows. At the attainments vector $(y_i, l_i)$, $i$ is deprived of level $\lambda_i = \lambda$ if $\lambda$ is the largest scalar for which $i$ finds $(y_i, l_i)$ at least as desirable as each $(y, l) \in IDC(\lambda)$.

**Priority among deprived.**

Priority among deprived individuals is introduced by requiring that society averses transfers of attainments from more deprived individuals to less deprived ones. Then, social deprivation is measured by the sum of opportunely transformed individual deprivations:

$$D(x_N) = \sum_{i \in N} f \circ \phi \circ \lambda_i (y_i, l_i),$$

where $f$ is the **priority function** and the function $\phi$ ensures that, for each $i \in N$, $\phi \circ \lambda_i (y_i, l_i)$ is a convex representation of individuals’ deprivation levels. As already clarified, the priority function plays a similar role as $f$ in the generalized mean income gap (5): it establishes how much society is concerned with the most deprived individuals. Following the income poverty,
I set $f$ to be the square function. The function $\phi$ is instead endogenous and requires no ethical choice.

**Empirical illustration.**

A crucial ingredient for the criterion is individuals’ preferences. The methodology adopted here is use observed labor supply choices to estimate the random utility model of Aaberge et al. (1999). The analysis generates more than 150 different types of individuals, depending on observables (age and education). Each type is associated a specific preference relation. Applying the above criterion to the estimated preferences gives a level of social deprivation (in per-capita terms) of $D = \frac{1}{n} \sum_{i \in N} [\lambda_i (y_i, l_i)]^2 = 0.0005$. For details, see Appendix A.

To shed light on the importance of responsibility for choices, note that the number of individuals identified as deprived (4.4 percent of the population) is larger than the number of individuals who earn less than 60 percent of the median income (2.9 percent). Two effects are in place. First, some individuals earn more than 60 percent of the median income, but since they work significantly more than median labor supply – while they would rather not – they are still considered as deprived. Second, some individuals earn less than 60 percent of the median income, but since they enjoy leisure “significantly” more than the median individual they are not considered deprived. These are those individuals that, even if offered, would not accept a job that would move them out of income poverty. In total, 2.4 percent of individuals are deprived according the preference-sensitive index adopted here, but are not according to income only. For 0.9 percent of individuals the converse holds.

5 Conclusions

The indices of individual and social deprivation characterized in this paper introduce a novel way to assess poverty. These indices: (i) are multidimensional; (ii) compare individuals by the set of attainments they are deprived of; (iii) are robust to measurement errors; (iv) express aversion to inequality
among the poor; and (v) respect individuals’ preferences. Moreover, such indices can account for differences in needs and extend to categorical dimensions of deprivation. The following table summarizes the main characteristics of the current proposal in relation to some of the well-known alternatives reviewed in Section 4 (For the FGT measure, the properties hold only when the index is convex).

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<tr>
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While overcoming several drawbacks of currently adopted indices, the results of this paper also emphasize that fundamental ethical choices are unavoidable and remain open to debate. These choices include the dimensions to take into account, the weights to attribute to each dimension, and the priority to place on the most deprived individuals. Importantly, such ethical choices necessarily depend on each specific application (i.e. absolute or relative poverty, local or global poverty), on the data available for the poverty assessment, and on the ethical views that society wants to embrace.
A Empirical exercise: details

A.1 Estimation of preferences

The database is the 2016 Norwegian register data, restricted to single men without children with age between 22 and 61. It consists of 90462 individuals. For each individual I use information about their after-tax income, labor supply, age, education, wage rate.

Each individual $i \in N$ is assumed to have preferences defined over after-tax income $y_i$, leisure $l_i$, and unobservable factors $q$ (i.e. non-pecuniary benefits related to the specific job). Income is measured in million NOK. Let $h_i$ denote the number of hours worked per year, the leisure is normalized as follows $l_i = 1 - \frac{h_i}{4160}$ ($l_i = 1$ corresponds to 80 hours of work per week). Preferences of individual $i$ can be represented by:

$$U_i(y, l, z) = v_i(y, l) \varepsilon(q),$$

which consists of an individual-specific “deterministic” component $v_i(y, l)$ and a common “stochastic” component $\varepsilon(q)$. Following Aaberge et al. (1999), assume that $\varepsilon(q)$ is extreme-value distributed of type III. Let the deterministic component of preferences be given by:

$$\ln v_i(y, l) = \left[\beta_i y^\zeta + (1 - \beta_i) l_i^\zeta\right]^\frac{1}{\zeta}.$$

Each individual is assigned a random opportunity set, consisting of triplets such as $(y, l, q)$. Each individual selects the alternative that maximizes her preferences $U_i$. Then, the parameters $\beta_i$ and $\zeta$ are identified to maximize the likelihood between the observed distribution of after-tax income and leisure pairs and the estimated ones. The random opportunity sets and the preference parameters $\beta_i$ are type specific and depend on the observable characteristics of individuals. Preferences depend on age, age squared, and 4 different
levels of education: this leads to 156 different types of individuals.\footnote{In the empirical application $\varepsilon(q)$ is set to unity and, thus, disregarded. I leave to future research the attempt to use the information on the unobservable component estimated for each individual to improve the measurement of individual and social deprivation.}

Formally, $\beta_i \equiv \frac{\beta_0}{\beta_0 + \delta_0 + \delta X_i}$, where $\beta_0 = 0.15$ and $\delta_0 = 0.75$ are constants and $\delta \in \mathbb{R}^5$ is a vector collecting the estimates for, in order, the multipliers for age (divided by 100) $\delta_1 = -1.21$, age squared (divided by 100$^2$) $\delta_2 = 1.88$, secondary school (1 if highest degree) $\delta_3 = -0.09$, college or university lower degree education $\delta_4 = -0.06$, and university higher degree education $\delta_5 = -0.18$. For simplicity, the elasticity of substitution between consumption and leisure is assumed to be equal across individuals and estimated as $\zeta = -2.55$. All coefficients are highly significant.\footnote{I report the t-values in parenthesis: $\beta_0$ (22.0); $\zeta$ (-91.4); $\delta_0$ (16.9); $\delta_1$ (-6.0); $\delta_2$ (7.8); $\delta_3$ (-12.0); $\delta_4$ (-6.9); and $\delta_5$ (-16.0).}

The interpretation is as follows, individuals’ willingness to work (or the relative importance given to consumption) first increases and then decreases with age, achieving a peak at age 32. As for education, the highest willingness to work is associated to the individuals with university higher degree education; the lowest is of those with no education; the individuals with highschool education are slightly more inclined to work than those with college or university lower degree education.

A.2 Individual and social deprivation

I first recall some definitions. The \textit{no-deprivation set} is:

$$NDS \equiv \left\{ (y, l) \in X \big| \left[ \beta y^\gamma + (1 - \beta) l^\gamma \right]^{\frac{1}{\gamma}} \geq \nu \right\},$$

where $\beta, \gamma, \nu$ are ethical parameters. The parameters are uniquely identified by the ethical choices discussed in Subsection 4.3. The \textit{iso-deprivation contour} of level $\lambda \in [0, 1]$ is the set:

$$IDC(\lambda) \equiv \left\{ (y, l) \in X \big| \left[ \beta y^\gamma + (1 - \beta) l^\gamma \right]^{\frac{1}{\gamma}} = (1 - \lambda) \nu \right\}.$$
Clearly, $IDC(0) = Fr \{NDS\}$ and $IDC(1) = \{0\}$. Consider individual $i \in N$ with preferences $R_i$. Her individual deprivation function $\lambda_i : X \to \mathbb{R}$ is defined by setting for each $(y,l) \in X$,

$$\lambda_i (y,l) \equiv \min \{\lambda \in [0,1] | (y,l) \ I_i (y',l') \text{ for some } (y',l') \in IDC(\lambda)\}.$$

By homotheticity of preferences and of the iso-deprivation contours, the above individual’s deprivation function is homogeneous of degree 1 and, thus, a least convex function. This allows disregarding the cardinalizing function “$\phi$,” said differently, for any ordinally equivalent representation of individual $i$’s deprivation, the endogenous cardinalizing function $\phi$ would have transformed these in an affine transformation of $\lambda_i$. Let $V_i \equiv \min_{(y,l) \in NDS} v_i (y,l)$ be the minimum level of well-being of $i$ when she is not deprived. By simple manipulation, the individual deprivation function can be rewritten as:

$$\lambda_i (y,l) = \max \left\{0, \frac{V_i - v_i (y,l)}{V_i} \right\}.$$

Social deprivation is measured by the sum of a convex transformation of the individuals’ deprivations. Using a power transformation with exponent 2, gives the following per capita index of social deprivation:

$$D(x_N) \equiv \frac{1}{n} \sum_i [\lambda_i (y_i,l_i)]^2.$$

B Proofs

B.1 Proof of Theorem 1

The proof that a deprivation ranking with a representation (1) satisfies the axioms is quite straightforward.

For each $i \in N$, $-\lambda_i$ is a non-decreasing transformation of a representation of preferences $R_i$: it is a representation of preferences whenever $i$ is

\footnote{By monotonicity of preferences, it is irrelevant whether the iso-deprivation contour of level 0 is the entire no-deprivation set or only its frontier.}
deprived (for each \( x_i \) such that \( \text{NDS} R_i x_i \)); it is constant whenever \( i \) is not deprived (for each \( x_i \) such that \( C \sim P_i x_i \)). Thus, weak Pareto follows. Since the functions \( f, \phi \), and, for each \( i \in N \), \( \lambda_i \) are continuous, the deprivation ranking \( \succeq \) is continuous. Since the representation of \( \succeq \) is additive over individuals, separability holds. By convexity of \( f \) and, for each \( i \in N \), \( \phi \circ \lambda_i \), equal-preference transfer follows.

Since each function \( \lambda_i \) is largest at the 0 attainment vector, first strictly decreasing and then constant, there exists a set \( C^u \) (for example, \( C = \text{NDS} \)) such that Condition (i) of no-deprivation fairness holds. Finally, assume that Condition (ii) of no-deprivation fairness is violated. Then, there exists a \( x_i^* \in C^u, i \in N, \) and \( \bar{x}_i \in X \) with \((\bar{x}_i, x_i^*) \succ x_i^* \), such that for each \( x_i \in X \) with \( x_i I_i \bar{x}_i \), each \( j \in N \setminus \{i\} \) and each \( x_j \in X \) with \( x_i \succ x_j \), it holds that \((x_j, x_j^*) \succeq (x_i, x_i^*) \). By construction of \( \lambda_i \), the index of deprivation of \( i \) at \( \bar{x}_i \) is the index \( \lambda_i \) associated to the smallest lower contour set of \( \succcurlyeq \) such that \( i \) finds each element of this set at least as desirable as \( \bar{x}_i \). Let \( x_i' \) be (one of) \( i \)'s least preferred bundles in this set (the existence is ensured by continuity of preferences and the assumption that \( \lim_{|x_i| \to \infty} u_i (x_i) = \infty \)). By continuity of preferences, \( x_i' I_i \bar{x}_i \). Moreover, each \( x_j \succ x_i' \) belongs to a smaller lower contour set of \( \succcurlyeq \). By construction, the index of deprivation associated to \( x_j \) is such that \( \lambda_j (x_j) < \lambda_i (\bar{x}_i) \). This contradicts \((x_j, x_j^*) \succeq (x_i, x_i^*) \) and proves that Condition (ii) holds.

I next show the reverse implication. The proof is divided in three steps.

**Step 1.** If a deprivation ranking \( \succeq \) satisfies weak Pareto, continuity, and separability, then there exists a continuous function \( D : X_N \to \mathbb{R} \), and, for each \( i \in N \), a non-increasing real-valued function \( g_i \), and a representation \( U_i : X \to \mathbb{R} \) of her preferences \( R_i \), such that for each pair \( x_N, x'_N \in X_N \), \( x_N \succeq x'_N \) if and only if

\[
D (x_N) \equiv \sum_{i \in N} g_i \circ U_i (x_i) \geq \sum_{i \in N} g_i \circ U_i (x'_i) \equiv D (x'_N).
\]
implies that deprivation fairness violates an attainment vector. The nuity of individual preferences and of the deprivation ranking, there exists a no-deprivation set $D \subset X$, a weak order $\succsim$ identifying the iso-deprivation contours, and a strictly increasing and continuous real-valued function $g$ such that, for each individual $i \in N$, $g_i \circ U_i = g \circ \lambda_i$, where $\lambda_i$ is defined in Section 2. That is, the deprivation ranking $\succeq$ can be represented by $D = \sum_{i \in N} g \circ \lambda_i$.

**Proof.** Deprivation fairness directly postulates (Condition (i)) the existence of a non-empty and closed set $C \subset X$ such that for each $i \in N$, each $x_n \in X_N$ with $x_i \in C$, and each $a_i \in X$ with $a_i \geq x_i$, $(0, x_{-i}) \succ (x_i, x_{-i}) \succsim (a_i, x_{-i})$. Thus, $0 \not\in C$. Let $x_i$ be (one of) the attainments vector(s) that $i$ finds least desirable in $C$; it’s existence follows from each $i$’s preferences admitting a strictly increasing and concave representation $u_i : X \to \mathbb{R}$ such that $\lim_{|x_i| \to \infty} u_i (x_i) = \infty$ and since $C$ is non-empty and closed. Then, using the representation of Step 1, $g_i \circ U_i (0) > g_i \circ U_i (x_i) = g_i \circ U_i (a_i)$ for each $a_i \geq x_i$ with $x_i \in C$. By monotonicity of preferences, $g_i \circ U_i (0) > g_i \circ U_i (x_i) = g_i \circ U_i (x_i)$ for each $x_i \in \mathbb{R}^n$.

Let $x_i^* \in C$. I show next that $(0, x_{-i}^*) \succeq (0, x_{-j}^*)$ for each $i, j \in N$. Assume not: then, without loss of generality, $(0, x_{-j}^*) \succ (0, x_{-i}^*)$. By continuity of individual preferences and of the deprivation ranking, there exists an attainment vector $\varepsilon_j \succ 0$ such that $(\varepsilon_j, x_{-j}^*) \succ (0, x_{-i}^*)$. This directly violates deprivation fairness (Condition (ii)), where $\bar{x}_i = 0$. This result also implies that $g_j \circ U_j (0) - g_j \circ U_j (x_{-i}) = g_j \circ U_j (0) - g_j \circ U_j (x_{-j})$ for each $i, j \in N$. 

**Step 2.** If the deprivation ranking $\succeq$ also satisfies deprivation fairness, then there exists a no-deprivation set $NDS \subset X$, a weak order $\succeq$ identifying the iso-deprivation contours, and a strictly increasing and continuous real-valued function $g$ such that, for each individual $i \in N$, $g_i \circ U_i = g \circ \lambda_i$.
Let $K \equiv [0, g_i \circ U_i (0) - g_i \circ U_i (\xi_i)]$. By continuity, for each $k \in K$ and each $i \in \mathbb{N}$, there exists an attainments vector $x_i (k)$ such that $g_i \circ U_i (x_i (k)) - g_i \circ U_i (\xi_i) = k$. It follows that $(x_i (k), \xi_i) \succeq (x_j (k), \xi_j)$ for each $i, j \in \mathbb{N}$. For each $i \in \mathbb{N}$, let the upper-contour sets at $x_i (k)$ be denoted $UCS_i (k) \equiv \{x_i \in X | x_i R_i x_i (k)\}$. The intersection of these upper-contour sets is $UCS (k) \equiv \bigcap_{i \in \mathbb{N}} UCS_i (k)$. Clearly, by definition of upper contour set, it cannot be that $x_i (k) P_i x$ for some $x \in UCS (k)$. Assume instead that, for each $x \in UCS (k)$, $x P_i x_i (k)$. By continuity of $U_i$, there exists an attainments vector $x_i^+ \in X$ such that, for each $x \in UCS (k)$, $x P_i x_i^+ P_i x_i (k)$ and $(x_j (k), \xi_j) \succ (x_i^+, \xi_i)$. Since $x_i^+ \notin UCS (k)$, for each $x_i I_i x_i^+$ there exists $j \neq i$ and $x_j \succ x_i$ with $x_j I_j x_j (k)$ and, by transitivity of the deprivation ranking, $(x_j, \xi_j) \succ (x_i, \xi_i)$. This is a violation of deprivation fairness (Condition ii). Thus, for each $i \in I$ and each $k \in K$, $x_i (k) I_i w_i$ with $w_i$ one of the least desirable attainment vectors for $i$ in $UCS (k)$. Moreover, for each $i, j \in \mathbb{N}$ and each $k, k' \in K$, $g_i \circ U_i (x_i (k)) \geq g_j \circ U_j (x_j (k'))$ if and only if $k \geq k'$.

I next construct the weak order $\succeq$ and the corresponding iso-deprivation contours. First, let $NDS$ be the convex hull of $UCS (0)$. This is non-empty, closed, and, by definition, convex; it is thus a no-deprivation set. For each $k \in K$, let $C (k)$ be the convex hull of $UCS (k)$. Note that by concavity of preferences, for each $i \in I$, $i$’s least desirable attainment vectors in $UCS (k)$ are the same as in $C (k)$. Define $\succeq$ by setting, for each $x, x' \in X$, $x \succeq x'$ if and only if $\max [k \in K | x \in C (k)] \geq \max [k \in K | x' \in C (k)]$. This weak order is continuous, strictly antimonotonic over $X \setminus C$ (i.e. for each $x, x' \in X \setminus C$, $x \geq x'$ implies that $x' \succ x$), constant over $C$ (i.e. for each $x, x' \in C$, $x \sim x'$), and, by convexity of individuals’ preferences, has convex lower contours.

Now, let $\lambda : X \rightarrow [0, 1]$ be a representation of $\succeq$ such that, without loss of generality, $\lambda (x) = 0$ for $x \in NDS$ and $\lambda (0) = 1$. Given $\lambda$, individual $i$’s deprivation function $\lambda_i$ is defined by setting for each $x_i \in X$, $\lambda_i (x_i) = \min \{\lambda (x) | x I_i x_i\}$.

Finally, for each $i, j \in \mathbb{N}$ and each $k, k' \in K$, $\lambda_i (x_i (k)) \geq \lambda_j (x_j (k'))$ if

\[\text{As an example, let } \alpha \equiv \min [\alpha \in \mathbb{R}^+ | \alpha 1 \in NDS]. \text{ Clearly, } \alpha > 0. \text{ Then, for each } x \in X, \text{ let } \lambda (x) = \frac{x - \alpha}{\alpha} \text{ if either } x = \alpha 1 \text{ or if } x \sim \alpha 1.\]

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and only if \( k \geq k' \), which holds if and only if \( g_i \circ U_i (x_i (k)) \geq g_j \circ U_j (x_j (k')) \).

Thus, there exists a real-valued and increasing function \( g \) such that \( g_i \circ U_i = g \circ \lambda_i \).

**Step 3.** If the deprivation ranking \( \succeq \) also satisfies equal-preference transfer, then there exists a priority function \( f \) and a least joint convex function \( \phi \) such that \( g = f \circ \phi \). That is, the deprivation ranking \( \succeq \) can be represented by \( D = \sum_{i \in N} f \circ \phi \circ \lambda_i \).

**Proof.** Let \( j, k \in N \) be such that \( R_j = R_k \equiv R_0 \). By Step 2, \( \lambda_j = \lambda_k \equiv \lambda_0 \). Let \( x_N, x'_N \in X^n \) be such that: (i) \( x'_j = x'_k = \frac{x_j + x_k}{2} \); (ii) for each \( i \in N \setminus \{j, k\} \), \( x_i = x'_i \). By equal-preference transfer, \( x \succeq x' \). By the previous steps, this is equivalent to

\[
g \circ \lambda_0 \left( \frac{x_j + x_k}{2} \right) \leq g \circ \lambda_0 (x_j) + g \circ \lambda_0 (x_k) \frac{2}{2}.
\]

Since this holds for each pair of attainments vectors of \( i \) and \( j \), \( g \circ \lambda_0 \) is convex. As the argument holds for each pair of individuals with the same preferences, for each \( i \in N \), \( g \circ \lambda_i \) is convex. Let \( \phi \) be a least joint convex transformation. Its existence follows from Lemma 1 of Piacquadio (2017) by noting that \(-\phi \) is least joint concave. Then, there exists a priority function \( f \) (continuous, increasing, and convex) such that \( g = f \circ \phi \) and \( \succeq \) can be represented by \( D = \sum_{i \in N} f \circ \phi \circ \lambda_i \).

**B.2 Proof of Theorem 2**

I only show that the axioms imply the deprivation index (2). I follow the same steps of the proof of Theorem 1.

Step 1 directly extends. If a deprivation ranking \( \succeq \) satisfies weak Pareto, continuity, and separability, then there exists a continuous function \( D : X_N \to \mathbb{R} \), and, for each \( \theta \in \Theta \) and each \( i \in N_\theta \), a non-increasing real-valued function \( g^\theta_i \) and a representation \( U^\theta_i : X \to \mathbb{R} \) of her preferences \( R_i \),
such that for each pair \(x_N, x'_N \in X_N\), \(x_N \succeq x'_N\) if and only if

\[
D(x_N) \equiv \sum_{\theta \in \Theta} \sum_{i \in N_\theta} g_i^\theta \circ U_i^\theta(x_i) \geq \sum_{\theta \in \Theta} \sum_{i \in N_\theta} g_i^\theta \circ U_i^\theta(x'_i) \equiv D(x'_N).
\]

With the next step, I also impose needs-adjusted deprivation fairness on the deprivation ranking \(\succeq\).

**Step 2.** If the deprivation ranking \(\succeq\) also satisfies needs-adjusted deprivation fairness, then there exists a constant \(\bar{\alpha} > 0\) (identifying the deprivation functions \(\lambda_\theta^\ast\) in Section 3) and a strictly increasing and continuous real-valued function \(g\) such that, for each \(\theta \in \Theta\) and each \(i \in N^\theta\), \(g_i^\theta \circ U_i^\theta = g \circ \lambda_i^\theta\). That is, the deprivation ranking \(\succeq\) can be represented by \(D = \sum_{\theta \in \Theta} \sum_{i \in N^\theta} g \circ \lambda_i^\theta\).

**Proof.** The first two conditions of needs-adjusted deprivation fairness are equivalent to those of deprivation fairness, except that the second holds only for equal-type individuals. Thus, repeating Step 2 of the proof of Theorem 1, for each \(\theta \in \Theta\), there exists a weak order \(\succeq^\theta\) on \(X\) (identifying the iso-deprivation contours) and a no-deprivation set \(NDS^\theta\) such that \(\succeq^\theta\) is continuous, strictly antimonotonic over \(X \setminus NDS^\theta\) (i.e. for each \(x, x' \in X \setminus NDS^\theta\), \(x \succeq x'\) implies that \(x' \succ x\)), constant over \(NDS^\theta\) (i.e. for each \(x, x' \in NDS^\theta\), \(x \sim x'\)), and has convex lower contours. For each \(\theta \in \Theta\), let \(\phi^\theta : X \to [0, 1]\) be a representation of \(\succeq^\theta\) such that, without loss of generality, \(\phi^\theta(x) = 0\) for \(x \in NDS^\theta\) and \(\phi^\theta(0) = 1\). For each \(\theta \in \Theta\) and each \(i \in N^\theta\), let \(\phi_i^\theta\) be such that \(\phi_i^\theta(x_i) = \min \{ \phi^\theta(x) | x I_i x_i \}\) and define:

\[
\alpha_i^\theta = \min \{ \alpha \in \mathbb{R}_+ | \alpha B^\theta R_i x_i \text{ for some } x_i \in X \text{ with } \phi_i^\theta(x_i) = 0 \}.
\]

I prove next that there exists \(\bar{\alpha}\) such that \(\alpha_i^\theta = \bar{\alpha}\) for each \(\theta \in \Theta\) and each \(i \in N^\theta\). Assume not, then there exist \(\theta, \theta' \in \Theta\), \(i \in N^\theta\), and \(j \in N^{\theta'}\) such that \(\alpha_i^\theta < \alpha_j^{\theta'}\). For each \(\alpha > 0\), let \(x_i(\alpha), x_j(\alpha) \in X\) be such that \(\alpha B^\theta I_i x_i(\alpha)\) and \(\alpha B^{\theta'} I_j x_j(\alpha)\). Let \(\alpha \in (\alpha_i^\theta, \alpha_j^{\theta'})\) and \(x_N^\alpha \in NDS^\alpha_N\). Since \(\alpha > \alpha_i^\theta\), \(\alpha B^\theta P_1 x_i(\alpha_i^\theta)\) and \(x_i(\alpha) P_1 x_i(\alpha_i^\theta)\). By definition of \(\phi_i^\theta\), \(\phi_i^\theta(x_i(\alpha)) = 0\) and, consequently, \((x_i(\alpha), x_N^\alpha) \sim x_N^\alpha\). Con-
versely, since $\alpha < \alpha_j'$, $\alpha B^{\theta'} P_j x_j (\alpha_j')$, \( \phi_j^{\theta'} (x_j (\alpha_j')) > \phi_j^{\theta'} (x_j (\alpha)) = 0 \), and \( (x_j (\alpha), x^*_j) \triangleright x^*_N \). Let $\alpha^- \in (\alpha_j', \alpha)$ and $\alpha^+ \in (\alpha, \alpha_j')$. By monotonicity of preferences, $\alpha B^{\theta'} P_j x_i (\alpha^-)$ and $\alpha B^{\theta'} P_j x_j (\alpha^+)$. By efficiency, \( (x_j (\alpha^+), x^*_j) \triangleright (x_i (\alpha^-), x^*_i) \). This is a contradiction of Condition (iii), proving the existence of $\bar{\alpha}$.

I next show that for each $\alpha, \alpha' \in [0, \bar{\alpha}]$ with $\alpha \leq \alpha'$, each pair $\theta, \theta' \in \Theta$, each $i \in N^\theta$, each $j \in N^\theta'$, and each $x_N \in NDS^\theta_N$, \( (x_i (\alpha), x^*_i) \triangleright (x_j (\alpha'), x^*_j) \). The same argument as above shows that the converse relation, i.e. \( (x_j (\alpha'), x^*_j) \triangleright (x_i (\alpha), x^*_i) \), leads to a contradiction of Condition (iii). For each $\theta \in \Theta$ and $i \in N^\theta$, define:

$$\lambda^\theta_i (x_i) = \max \left\{ 0, 1 - \frac{\alpha}{\alpha} | x_i I_i x \text{ for some } x \in \alpha B^\theta \right\} .$$

Finally, for each pair $\theta, \theta' \in \Theta$, each $i \in N^\theta$, each $j \in N^\theta'$, and each $\alpha, \alpha' \in [0, \bar{\alpha}]$, \( \lambda^\theta_i (x_i (\alpha)) \geq \lambda^\theta_j (x_j (\alpha')) \) if and only if $\alpha \leq \alpha'$. By Step 1, \( \lambda^\theta_i (x_i (\alpha)) \geq \lambda^\theta_i (x_j (\alpha')) \) if and only if $g^\theta_i \circ U^\theta_i (x_i (\alpha)) \geq g^\theta_j \circ U^\theta_j (x_j (\alpha'))$. Thus, there exists a real-valued and increasing function $g$ such that $g^\theta_i \circ U^\theta_i = g \circ \lambda^\theta_i$ and $\triangleright$ can be represented by $D = \sum_{\theta \in \Theta} \sum_{i \in N^\theta} g \circ \lambda^\theta_i$.

**Step 3.** If the deprivation ranking $\triangleright$ also satisfies equal-preference transfer, then there exists a priority function $f$ and a least joint convex function $\phi$ such that $g = f \circ \phi$. That is, the deprivation ranking $\triangleright$ can be represented by $D = \sum_{\theta \in \Theta} \sum_{i \in N^\theta} f \circ \phi \circ \lambda^\theta_i$.

**Proof.** See the proof of the corresponding step of Th.1. 

**B.3 Proof of Theorem 3**

Again, the proof that the ranking satisfies the axioms is omitted.

I prove the converse implication. By deprivation fairness$^+$, there exists a vector of categorical attainments $s^* \in S$ such that, if each $i$’s attainment is $s_i = s^*$, the same conditions in deprivation fairness are imposed. Thus, I first focus on $X^+_N$ such that $s_N = s^*_N$. All the axioms in Theorem 1 hold on
this space. Thus, the deprivation ranking \( \succeq \) (on \( X_N^+ \) with \( s_N = s_N^* \)) can be represented by \( D = \sum_{i \in N} f \circ \phi \circ \tilde{\lambda}_i \), where \( \tilde{\lambda}_i \) is defined in Section 3.

I next extend the representation to \( X_N^+ \). By assumption, for each \( x_N^+ \in X_N^+ \) there exists a social state \( \bar{x}_N^+ \in X_N^+ \) with \( s_N = s_N^* \) such that, for each \( i \in N, (x_i, s_i) \sim_I (\bar{x}_i, s^*) \). By weak Pareto\(^+\) and continuity\(^+\), \( x_N^+ \succeq \bar{x}_N^+ \). Thus, the result follows.

References


