**Tax deferral and investment incentives: the optimal design of a tax-deductible reserve**

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**Abstract**

Several countries have implemented tax deductible reserves, partly to incentivize business investment. Investment reserves are designed directly for this purpose, whereas tax allocation reserves defer taxes generally and may serve other goals as well. This paper employs a delayed-response optimal control model to study how various forms of tax-deductible reserves affect investment incentives. The effects are observed to vary depending on the type and design of the reserve. Tax allocation reserve lowers the effective tax rate and produces a small reduction in the firm’s cost of capital. The impacts of investment reserves are more complex. With a low non-investment penalty and a low ceiling for allocations the investment reserve works similarly to a tax allocation reserve. Instead, with a high penalty and a high ceiling it comes close to a neutral cash flow tax. We further analyze the socially optimal design of investment reserves and find that ... [work in progress].

**JEL classification**: C61, D25, H25, H32

**Keywords**: corporate tax; investment; tax deferral; reserve; provision; firm behavior

1. **Introduction**

Economic analysis suggests that investment and capital stock are important determinants of productivity and welfare of a country. Governments have indeed searched for efficient ways to spur business investment, corporate taxation being one frequently used means for this purpose: most OECD countries have cut their tax rates over the last few decades, numerous countries have provided accelerated depreciation schemes for equipment and structures, and some others have allowed the costs of equity to be deductible in corporate taxation (e.g. Belgium, Italy and Turkey). ¹

In this study we consider a further category of measures to affect investment conditions, namely tax-deductible reserves. These provisions bring forward future expenses and therefore defer tax payments to later periods. The resulting decline in the present value of taxes tends to strengthen the incentives for investment and other economic activities. While there are various forms of tax-deductible reserves (and provisions) serving many

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¹ Recent literature has provided evidence of the effect of tax incentives on corporate investment; see e.g. Bond and Xing (2015), Zwick and Mahon (2017), Ohrn (2018) and Maffini et al. (2019).
distinct purposes, our focus here is on an investment reserve and a tax allocation reserve.

The investment reserve allows the firm to allocate a share of its pre-tax profit to a reserve and deduct the allocated amount from its taxable income. The firm may then cover new investment from the allocated tax-exempt funds. The amount that is not used on investment will be returned as taxable income.

Under the tax allocation reserve, the firm may allocate a share of its pre-tax profit to the tax-deductible reserve. Now, however, the allocated amount cannot be used to cover investment but, rather, it must be returned as taxable income within a predetermined period. In our analysis the tax allocation reserve can thus be seen as a special case of the investment reserve with no investment component.

Applications of both types can be found in the OECD countries. Sweden has a tax allocation reserve under which the firm can allocate up to 25 per cent of its pre-tax profit to the reserve for a maximum time of 6 years (Deloitte, 2016). Investment reserves, which were important elements of the Nordic corporate tax systems until the early 1990’s, are currently operated at least in Finland and Germany, mostly to moderate the tax burden of small firms. In Finland, recently, proposals have been made for reintroducing a more general investment reserve to encourage private investment.

The goal of our study is to analyze the investment implications of different reserve forms and, also, how such a reserve should be designed to satisfy society’s goals concerning efficiency and tax revenue. Previous literature has argued that the investment consequences may be sensitive to the reserve design. Södersten (1989) and Auerbach et. al. (1995) model Sweden’s previous “investeringsfond” system and detect several distinct regimes depending e.g. on the size of the allocation ceiling. They particularly show that in some special cases the system may have a strong impact on incentives but using more moderate levels of parameter values the reserve tends to have a smaller effect on the average tax rate of the firm with a limited impact on investment incentives.

Kari (2017) analyses a “modern” version of an investment reserve where the firm can choose between using the reserve to cover investment and bringing it back to taxable profit. The analysis suggests that - from the efficiency point of view - there should be some mechanism, stick or carrot, to encourage the firm to using the reserve on investments. The

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2 Tax legislation commonly allows other forms of tax-deductible reserves and provisions some of which are intended to target expected future expenses or losses. Examples of these are bad debt reserve and provision for guarantee obligations.

3 Other terms of this reserve type used in the literature are “profit reserve” (ZEW, 2015) and “periodization fund/reserve” (Lindhe, 2002; Kari 2017).

4 A tax allocation reserve is sometimes seen to have desirable effects since it allows firms to level off the variation in taxable income (and therefore complement existing tax loss offset means) and reduces tax-induced distortions by cutting the effective tax rate.

5 For a review of the previous investment reserve systems in Denmark, Finland, Norway and Sweden, see Andersson et al. (1998). The systems were originally aimed at being tools of counter-cyclical fiscal policy but were later transformed to general measures to promote business investment. An important aspect of the Finnish and Swedish systems was the requirement that to be able to deduct the allocation from taxable income, the firm had to deposit a fraction of the allocated amount in an interest-free blocked account at the central bank. In the recession years the government permitted firms to withdraw funds from these accounts (and use the reserve) to finance investment (Agell et al., 1995; Andersson et al. 1998).

6 Also, Belgium has applied an investment reserve but abolished it in the recent corporate tax reform enacted in 2017.

7 Such proposals were evaluated by the Expert group on business taxation (2017).
study focuses on a variant of such a tool, where a penalty is levied if the allocated funds are not used appropriately. 8

Our study extends the literature on tax-deductible reserves in several ways. First, we contribute by modeling the deferral effect using a delayed-response optimal control model. This approach has the advantage that it can accurately account for the delay in tax payments. Previous literature has either just assumed the form of the effect on the firm’s cost of capital (Lindhe, 2002; ZEW, 2015) or has used a distributed lags model, where the delay is dispersed over time following an exponential distribution (Kari, 2017). Second, we analyze the optimal policy, not just in the steady state, but over the firm’s life cycle. This provides a richer view of firm-level effects and provides important input for the welfare analysis. Finally, and most importantly, we address the question of how the incentive mechanism included in a “modern” investment reserve should be designed to satisfy the government’s welfare goals. In this analysis we apply a welfare function that is a combination of the representative household’s corporate source income and the government’s corporate tax revenue. This allows us to consider the tradeoff between the benefits of improved investment conditions and the costs in the form of lost revenue due to tax incentive provisions.

Our results confirm that the investment effects of a tax-deductible reserve vary depending on the type of the reserve and the values of several parameters such as size of the ceiling, the rate of the penalty factor and the length of the deferral period. In particular, we find that with a low non-investment penalty and a low ceiling for allocations the investment reserve affects like a tax allocation reserve. Instead, with a high penalty and a high ceiling it comes close to a neutral cash flow tax. In the welfare analysis we make a comparison between different reserve designs while keeping the standard corporate tax and the cash flow tax as benchmarks.

The paper proceeds as follows. In Sec. 2 we set up the investment model and illustrate the firm’s optimal policies under different reserve designs. Sec. 3 provides the welfare analysis and Sec 4 discusses and concludes. Details of the analysis are provided in Appendix A.

2. The firm’s investment decisions under tax-deductible reserves

2.1 The model

This section sets up a dynamic investment model to study the investment incentives provided by different types of reserves. It considers the dynamics of an equity-financed value-maximizing firm in continuous-time. The firm produces output with homogenous capital \( K \) and finances its operations through profits \( \pi(K) \). The firm spends its resources on dividends \( D \), investment \( I \) and corporate taxes \( T \), yielding to the budget constraint

\[
\pi(K) = D + I + T.
\]

The capital stock \( K \) depreciates at rate \( \delta \in (0,1) \) and its equation of motion is given by

\[
\dot{K} = I - \delta K.
\]

We analyze two types of tax-deductible reserves, a tax allocation reserve and an investment reserve. In both reserve types, the firm may allocate a tax-deductible amount \( C \) of pre-tax

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8 The study roughly follows the lines of the previous Finnish investment reserve system, where a mark-up of 30 % was added to the firm’s taxable income if the funds were not used on investment but rather entered as taxable income.

9 The profit function \( \pi \) is strictly concave on positive real line \( \mathbb{R}_+ \) and satisfies the usual Inada conditions.
profit to the reserve. This amount is constrained by a share, \( m \in [0, 1] \), of profit after depreciation, \( C \leq m(\pi(K) - aB) \), where \( a \in [\delta, 1] \) is the rate of fiscal depreciation and \( B \) is the accounting stock of capital.

In the case of a tax allocation reserve, the amount \( C \) must be returned as taxable income within a predetermined period \( v \). As deferring taxes is profitable, an optimizing firm always chooses the maximum time of deferral. Thus, at time \( t \), the amount returned to taxable income equals \( C_v(t) = C(t - v) \). In the case of investment reserve, the amount \( C \) (or some part of it) can instead be used to cover new investment. For this amount, the firm cannot claim depreciation allowances, since investment covered this way is regarded as fully written off for tax purposes. Hence, the firm only adds the investment net of this amount to its depreciable assets. We denote the part of \( C \) used to cover investment by \( G \). With this notation, the equation of motion for the accounting stock of capital becomes

\[
\dot{B} = I - G - aB.
\]

Hence \( B \) increases with new investment \( I \) net of any amount \( G \) covered from the reserve and decreases with fiscal depreciation \( aB \).

The firm’s tax bill is given by

\[
T = \tau[\pi(K) - aB - (C - z(C_v - G_v))],
\]

where \( \tau \in (0,1) \) is the corporate tax rate, and the element in brackets is taxable income. The latter is calculated as operating profit \( \pi(K) \) less fiscal depreciation allowances \( aB \) and allocations \( C \) to the reserve net of the amount \( z(C_v - G_v) \) entered as taxable income. Here \( G_v(t) = G(t - v) \). By the constant \( z \geq 1 \) we model a tax penalty (stick), levied for the part of the reserve that is not used against investment but rather entered into taxable income. If \( z = 1 \), there is no penalty, and if \( z > 1 \), the system includes a penalty. Hence, in the model, \( z \) represents an incentive mechanism through which the government can adjust the firm’s willingness for using the reserve on investment.

We apply the constraints \( 0 \leq C \leq m(\pi(K) - \delta K), 0 \leq G \leq I, G \leq C \) and \( K(0) = K_0 \geq 0 \), where \( K_0 \) is the endogenous initial stock of capital.

The objective of the firm is to maximize the net value of its shares

\[
V = \int_0^\infty De^{-\rho t}dt - K_0,
\]

subject to the above constraints and definitions, where \( \rho \) is the time-invariant rate of interest at which the economic agents can borrow and lend in the capital markets.

2.2 The firm’s optimal policy

This section considers the firm’s investment decisions under different reserve designs. The analysis splits into five distinct cases with different implications on investment. The standard corporate tax and the cash flow tax are held as benchmarks.

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10 In the case of a tax allocation reserve, \( G \) does not exist and can be interpreted as taking value zero, \( G = 0 \).

11 The model assumes that the use of the reserve on investment, \( G \), occurs at the same moment the funds are allocated. Hence there is no delay between the allocation and the use of the reserve. This assumption is made to simplify the model. The assumption is supported by Kari (2017), where this restriction can be avoided due to a slightly simpler framework; the solutions of the investment model are similar.
Box 1 summarizes solutions of this model in various cases of interest. The proofs of these cases as well as further technical assumptions are given in Appendix A. The following abbreviations are applied: \( c = e^{-v \rho}, A = \frac{a}{a + \rho}, A' = \frac{a(1-m)}{a(1-cm) + \rho}, e_1 = \tau(1 - m + cm) \) and \( e_2 = \tau(1 - m + A' cm) \). \(^{12}\)

<table>
<thead>
<tr>
<th>Box 1: Main findings from the investment model</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Case (1): Benchmark (no reserve; ( C = G = 0, m = 0 ))</strong></td>
</tr>
<tr>
<td>- Applicable parameter range: all</td>
</tr>
<tr>
<td>- Cost of capital: ( \pi'(K_1) = \frac{\rho + \delta}{1 - \tau} (1 - \tau A) )</td>
</tr>
<tr>
<td>- Corporate income tax: ( T = \tau(\pi(K_1) - aB_1) ), where ( B_1(t) = \frac{K_1((\alpha - \delta)e^{-a\tau + \delta})}{\alpha} )</td>
</tr>
<tr>
<td>- As a special case of ( A = 1 ) we get the cash flow tax, where ( \pi'(K_1) = \rho + \delta )</td>
</tr>
</tbody>
</table>

| **Case (2): Tax allocation reserve (\( G = 0 \))** |
| - Applicable parameter range: all |
| - Cost of capital: \( \pi'(K_2) = \frac{\rho + \delta}{1 - e_1 A} (1 - \tau A) \) |
| - Corporate income tax: \( T = \tau[(1 - m)(\pi(K_2) - aB_2) + zm(\pi(K_2) - aB_2_v)] \), where \( B_2(t) = \frac{K_2((\alpha - \delta)e^{-a\tau + \delta})}{\alpha} \) |

| **Case (3): Investment reserve, low penalty.** |
| - Applicable parameter range: \( cz < e_1 A/\tau \) |
| - The formulas for cost of capital and corporate income tax are identical to Case 2. |

| **Case (4): Investment reserve, low reserve upper bound, medium or high penalty.** |
| - Applicable parameter range: low value of \( m, cz > e_2 A/\tau \) |
| - Cost of capital: \( \pi'(K_4) = \frac{\rho + \delta}{1 - e_2 A} (1 - \tau A) \) |
| - Corporate income tax: \( T = \tau(1 - m)\pi(K_4) - aB_4 \), where \( B_4(t) = K_4 e^{-a(1-m)t} + \frac{\delta K_4 - m\pi(K_4)}{a(1-m)} (1 - e^{-a(1-m)t}) \). |

| **Case (5): Investment reserve, high reserve upper bound, medium penalty \( z \).** |
| - Applicable parameter range: high value of \( m, e_2 A/\tau \leq cz < 1 \) |
| - Cost of capital (steady state): \( \pi'(K_5) = \rho + \delta \frac{1 - e_2 A}{1 - e_1 A} \) |
| - Corporate income tax (steady state): \( T = \tau[(1 - m + z)\pi(K_5) - (1 - m) aB_5 - zmB_5 - z\delta K_5] \), where \( B_5(t) = K_5 e^{-at} \). |

| **Case (6): Investment reserve, high reserve upper bound \( m \), high penalty \( z \).** |
| - Applicable parameter range: high value of \( m, cz > 1 \) |
| - Cost of capital (steady state): \( \pi'(K_6) = \rho + \delta \) |
| - Corporate income tax: there is \( t^* > 0 \) such that \( T = \tau(1 - m)(\pi(K_6) - aB_6) \) for \( 0 < t < t^* \) and \( T = \tau(\pi(K_6) - aB_6 - \delta K_6) \) for \( t > t^* \), where \( B_6(t) = K_6(0)e^{-at} \). |

We next discuss some properties of these cases. We start with Case 1, which describes the benchmark case where no tax-deductible reserves are available to the firm. The optimal

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\(^{12}\) For tax allocation reserve \( z = 1 \); for investment reserve \( z \geq 1 \).
condition for the capital stock is
\[\pi'(K_1) = \frac{\rho + \delta}{1 - \tau} (1 - \tau A),\]
where \(A = \frac{\alpha}{\delta \rho}\) is the present value of fiscal depreciation allowances on a unit of capital calculated at the firm’s discount rate \(\rho\). This formula corresponds to the standard expression for the cost of capital assuming that the firm finances from retained earnings (King & Fullerton 1984; Deveraux & Griffith 2003). Corporate taxation affects the investments here through two channels: first, it reduces the return on marginal investment, reflected by the term \(1 - \tau\) in the denominator, and second, it reduces the cost of investment \(1 - \tau A\) in the numerator. In the special case of immediate expensing of investment, we would obtain \(A = 1\). When we substitute this into the formula for cost of capital, the tax term disappears. This demonstrates that corporation tax has no effect on the cost of capital under a cash flow tax. The intuition is that immediate expensing exempts the marginal return (normal return) on capital from taxes and tax only falls on economic rents.

Case 2 describes a tax allocation reserve, where the only way to release the funds in the reserve is to return them as taxable income. This means that the reserve cannot be used to cover investment expenditure \((G = 0)\). In this case, the firm always allocates the maximum amount into the reserve. The cost of capital becomes
\[\pi'(K_2) = \frac{\rho + \delta}{1 - e_1} (1 - e_1 A).\]
The difference to Case 1 is that the legal tax rate \(\tau\) is now substituted by the effective corporate tax rate \(e_1 = \tau (1 - m + cm)\), where \(c = e^{-v'}\) is the present value of one unit of deferred profit (we assume \(z = 1\) for simplicity). We suggest the following economic interpretation for \(e_1\): as \(c\) is the present value of deferred profit, \(e_1\) gives the weighted average of instantly paid tax and the present value of delayed taxes, with the share of profit taxed instantly, \(1 - m\), and the share for which taxation is delayed, \(m\), as weights. The effective tax rate can be rewritten as \(e_1 = \tau - \tau m (1 - c) < \tau\). The relevant tax rate that affects the returns and costs of investment is thus the effective rate, which is strictly lower than the statutory tax rate. This implies that the cost of capital is scaled down compared to the benchmark case (Case 1): \(\pi'(K_2) < \pi'(K_1)\). To our knowledge, the above formula for the cost of capital has not been derived in an optimization framework before,\(^\text{13}\) but it is still familiar from some previous studies. ZEW (2015) and Lindhe (2002) consider the effective corporate tax rate under the Swedish tax allocation reserve using a heuristic approach and end up to the same formula. Hence, our analysis provides a justification for the way these studies model the Swedish tax allocation reserve.

Cases 3 to 6 describe different cases of investment reserves, where the firm may choose between using (part \(G\) of) the new reserve \(C\) to cover investment and returning the reserve (or the remaining part \(C - G\) of it) after time \(v\) as taxable income. When the firm considers this choice, it weighs between lost tax savings from depreciation allowances (cost) and the penalty payment that it can avoid by using the reserve on investment (benefit). If the latter is higher the firm uses the reserve on investment and vice versa. We will see that the choices may differ radically between different cases.

Consider first Case 3, which is effective when the penalty rate is sufficiently low. Due to low penalty, the firm allocates the maximum amount to the reserve \((C = m(\pi(K) - aB))\) and no share of the reserve is used against investment \((G = 0)\). In this case the optimal solution looks the same as in Case 2 (tax allocation reserve). In particular, we have \(\pi'(K_3) = \frac{\rho + \delta}{1 - e_1} (1 - e_1 A)\).

\(^{13}\) Except in a slightly different form in Kari (2017).
Case 4 describes a situation, where the penalty $z$ is sufficiently high (medium high) so that the firm starts using the reserve to cover investment. However, the reserve upper bound $m$ is so small that the reserve maximum size is not enough to cover all the investment. The firm again allocates the maximum amount into the reserve: $C = m(\pi(K) - aB)$. The firm also can, and chooses to, invest all available reserve: $G = C$. The formula for the cost of capital becomes $\pi'(K_4) = \frac{\rho + \delta}{1-e_2} (1-e_2A)$, where $e_2 = \tau(1-m + A'm)$ and $A' = \frac{a(1-m)}{\rho(1-cm) + \rho'}$. We have $e_2 < e_1$, so that the cost of capital is strictly lower than in the case of a tax allocation reserve, $\pi'(K_4) < \pi'(K_2)$. Because all the reserve is used to cover investment, no penalty is due. Therefore, the cost of capital is independent of the penalty factor $z$. The new effective tax rate $e_2$ is a weighted average of the statutory tax rate and the increased tax due to the loss of regular depreciation allowances. As under the tax allocation reserve, the effect on investment is again channeled through a lowered effective corporate tax rate. This reduction in tax burden translates into a slightly lowered cost of capital (compared to standard corporate tax). 14

In Case 5, the reserve upper bound $m$ is high, but the medium-sized penalty $z$ is still sufficiently low ($cz < 1$) to encourage the firm to allocate the maximum amount into the reserve: $C = m(\pi(K) - aB)$. However, the reserve is now large enough for covering all investment from the reserve. Hence the firm chooses $G = I$. The cost of capital is now $\pi'(K_4) = (\rho + \delta)\frac{1-cz}{1-e_1}$. Taxation affects costs and returns asymmetrically. Net returns are affected by the effective tax rate $e_1 = \tau(1-m + czm)$. This is apparently related to the fact that the firm allocates the maximum amount to the reserve. This draws the firm’s average tax rate to $e_1$. This is also the effective tax rate faced by the marginal investment project.

The cost of investment is in turn reduced by the term $czt$, which is independent of depreciation allowances. This occurs because investment expenditure is covered from the reserve and is not expensed through the ordinary fiscal depreciation system. The apparent explanation for the term $czt$ is the following: In the present case the firm allocates the maximum amount to the reserve and spends a fraction of it to cover (all) investment expenditure. The rest will be returned to taxable income with a penalty. In this situation, each additional unit of investment reduces the amount that will be returned to taxable income and, therefore, saves the firm an amount of $czt$. As a result, the effective cost of marginal investment becomes $1 - czt$. 15

We finally consider Case 6, where both the reserve upper bound $m$ and the penalty factor $z$ are high ($cz > 1$). Due to the high values of $z$ and $m$ the firm has both an incentive and the resources to cover all new investment from the reserve. However, in contrast to Case 5, the firm does not allocate the maximum amount to the reserve. Due to the high penalty it only allocates the amount that it can use on investment, $C = G = I$. Following this policy, no penalty at the high rate will be due. In this case the cost of capital is $\pi'(K_6) = \rho + \delta$. Hence, under a high penalty and a high allocation ceiling, taxation has no effect on investment.

14 Our Case 4 is analogous to regime 3 in Södersten (1989), which he called the “new view” of the Swedish “investeringsfond” system. The key argument was that since the effects work through the (slightly lowered) effective corporate tax rate, their magnitude might be relatively small. This was in contrast to results in earlier literature, which claimed that the effects were large.

15 Our Case 5 is close to regime 2 in Södersten (1989), which was called there the “old view” of the “investeringsfond” system.
The apparent explanation for this neutrality result is that the system works like immediate expensing under a cash flow tax. The firm’s policy is to match its allocations to the level of its investment. This means that a one unit increase in investment is accompanied by a one-unit increase in tax-deductible allocations and by a tax saving of \( \tau \). As a result, the effective cost of investment is \( 1 - \tau \) as under a cash flow tax. Since both the costs and returns face the same tax rate \( \tau \), the effects of taxation cancel out.

In Case 6, the assumed “modern” investment reserve entails some clear ingenuity. The assumed high penalty means that the firm is not willing to allocate more than is used against new investment. Together with the high ceiling this means that the firm possesses underutilized tax allowances. This has important implications: if the firm decides to increase its investments by one further unit it can also increase the tax-deductible allocation to the reserve by the same amount. This creates the analogy to immediate expensing. The outcome particularly implies that, in this case, the reserve is not just a general tax allowance, which reduces the average tax rate on future and past projects, but rather is a targeted means to abolish the tax burden on marginal investment.

The model considers the firm’s optimal policy over its entire life cycle. It assumes that at time \( t = 0 \) an optimal amount of initial capital, \( K_0 \), is injected in the firm.\(^{16}\) If \( K_0 \) deviates from the steady state stock of capital, \( K^* \), the firm’s life cycle includes a phase where the capital stock either grows or contracts to the steady state size. We indeed find that in two of our cases, in Case 5 and Case 6, the life cycle includes a growth phase where the firm gradually increases its capital stock from a low value of \( K_0 \) to \( K^* \). In all other cases the firm invests its capital stock up to its steady state size already at the start-up moment \((t = 0)\). More information of the growth paths is given in Appendix A.

We finally consider a numerical example of the steady state cost of capital. We assume an investment, which is depreciated using the declining balance method. The rate of economic depreciation is \( \delta = 0.2 \) and the rate of fiscal depreciation is \( \alpha = 0.25 \). The statutory tax rate is \( \tau = 0.2 \). The maximum share of profit that can be allocated to a reserve is \( m = 0.25 \). The deferral time of the reserve is 3 years; \( v = 3 \). The interest rate is \( \rho = 0.05 \). With these assumptions the effective tax rates are \( e_1 = 0.193 \) and \( e_2 = 0.183 \), when there is no penalty factor; \( z = 1 \). The following table summarizes mathematical values of optimal cost of capital formulas in different cases. Observe however that all cases are not simultaneously feasible because of the inapplicable size of the penalty \( z \).

<table>
<thead>
<tr>
<th>Tax system</th>
<th>Cost of capital, low penalty ((z = 1))</th>
<th>Cost of capital, high penalty ((z = 1.5))</th>
<th>Cost of capital, cut-off penalty ((z = 1/c))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow tax</td>
<td>5 %</td>
<td>5 %</td>
<td>5 %</td>
</tr>
<tr>
<td>Case 1</td>
<td>6.04 %</td>
<td>6.04 %</td>
<td>6.04 %</td>
</tr>
<tr>
<td>Case 2</td>
<td>6.00 %</td>
<td>(6.14 %; case not valid)</td>
<td>6.04 %</td>
</tr>
<tr>
<td>Case 3</td>
<td>6.00 %</td>
<td>6.14 %</td>
<td>6.04 %</td>
</tr>
<tr>
<td>Case 4</td>
<td>5.93 %</td>
<td>5.93 %</td>
<td>5.93 %</td>
</tr>
<tr>
<td>Case 5</td>
<td>5.65 %</td>
<td>(3.61 %; case not valid)</td>
<td>5.00 %</td>
</tr>
<tr>
<td>Case 6</td>
<td>(5.00 %; case not valid)</td>
<td>5.00 %</td>
<td>5.00 %</td>
</tr>
</tbody>
</table>

Note that only Cases 2, 3 and 5 depend on the value of \( z \).

\(^{16}\) We assume there are no constraints in financing the start-up stock of capital.
3. The optimal design of an investment reserve

3.1 The Government’s problem
In this section, we consider how a welfare maximizing government should design the investment reserve system and whether an optimally designed reserve model brings a higher welfare than the benchmark systems, standard corporate tax and corporate cash flow tax.

Social welfare is defined as a linear combination of corporate tax revenue ($T$) and net-of-tax profits ($\pi^{AT}$), with a relative weight $\lambda \in [0,1]$:

$$W = T + \lambda \pi^{AT}$$

The net-of-tax profits are calculated as tax rate times the tax base, which is the gross profit $\pi(K)$ minus the economic depreciation ($\delta K$), the opportunity cost for capital ($\rho K$) and taxes ($T$).

$$\pi^{AT} = \pi(K) - \delta K - \rho K - T$$

Corporate taxes are calculated by multiplying the tax base with tax rate. Corporate taxes are like in the previous section (see eq. XXX), but now the fiscal depreciation is aligned with economic depreciation ($\alpha = \delta$). $\tau$ stands for the tax rate, $B$ for the book capital ($\dot{B} = I - G - \delta B$), $\alpha$ for the fiscal depreciation, $C$ for the tax-deductible reserve, $C_v$ for the return to taxable income, $G$ to investments made from the reserve and $z$ for the penalty.

$$T = \tau [\pi(K) - \delta B - (C - z(C_v - G))]$$

The government’s goal is to maximize $\varphi$, which is the present value of welfare flow $W$ discounted at $\rho$:

$$\varphi = \int_0^\infty W e^{\rho t} dt.$$ 

3.2 Optimal investment reserve

[work in progress]
4. Discussion and conclusions

We have studied two main forms of tax-deductible reserves and their effects on investment incentives. The first of them, a tax allocation reserve, is designed to allow the firm to defer the taxation of a fraction of its pre-tax profit. Also an investment reserve facilitates deferral but its apparent aim is to provide a tax-free source for financing investment. Therefore, the firm is expected to cover investment expenditure from the funds in the reserve. In our model the funds in the investment reserve may alternatively be released by entering them as taxable income. An essential element of the investment reserve is a penalty levied if the reserve is not used according to its main aim. The purpose of this element is to provide incentives for using the reserve on investment.

[work in progress]
References


Appendix A: proofs of the investment model cases in Box 1 (work in progress)

In this Appendix we prove the solutions of dynamic investment model cases from Section 2. We first restate the optimization problem with necessary assumptions and then prove each case separately. Let \( \mathbb{R}_+ = [0, \infty) \) and denote by \( \Omega \) the set of functions \( f \) on \( \mathbb{R}_+ \) such that \( f \) is almost everywhere continuous and \( \int_0^\infty |f(t)|e^{-\rho t} \, dt < \infty. \) Let \( 0 < \tau < 1, \rho > 0, \delta > 0, \alpha \geq \delta, \nu \geq 0, 0 \leq m \leq 1, z \geq 1, C_0 \geq 0 \) be constants. Abbreviate \( E(t) = e^{-\rho t}, C_v(t) = C(t - \nu) \) for \( t \geq \nu, C_v(t) = C(0) \) for \( 0 < t < \nu \) and \( c = e^{-\rho \nu} \). The objective is to find a policy \( (D, C, G, K_0) \in \Omega^3 \times (0, \infty), \) which maximizes

\[
\int_0^\infty D(t)e^{-\rho t} \, dt - K_0 \tag{A.1}
\]

subject to

\[
\pi(K) = D + I + T, \\
T = \tau[\pi(K) - aB - (C - \nu(C_v - G_v))], \\
\dot{K} = I - \delta K \quad \text{a.e. on } \mathbb{R}_+, \\
\dot{B} = I - G - aB \quad \text{a.e. on } \mathbb{R}_+, \\
0 \leq C \leq m(\pi(K) - aB), \\
0 \leq G \leq I, \\
G \leq C_v, \\
B(0) = K(0) = K_0 \geq 0, \\
C(0) = C_0,
\]

where (i) \( K \) and \( B \) are continuous and almost everywhere differentiable on \( \mathbb{R}_+ \), (ii) \( \pi(K), B, \) and \( G \) belong to \( \Omega \), (iii) \( \lim_{t \to \infty} K(t)e^{-\rho t} = 0 \), and (iv) \( \pi: \mathbb{R}_+ \to \mathbb{R}_+ \) satisfies the usual Inada conditions. A policy \( (D, C, G, K_0) \in \Omega^3 \times (0, \infty) \) is said to be admissible if it satisfies all of the above conditions. We denote the set of all admissible policies by \( \Gamma. \)

Case 1: Observe that Case 1 follows from Case 2, or from Case 3, by setting \( m = 0. \)

We next consider Case 2. Assume hence that \( G = 0 \) and \( \nu < 1. \) Let \( h = \tau(1 - m(1 - cz)) \) and define the number \( K^* > 0 \) by

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17 A function \( f \) is almost everywhere (a.e.) continuous (respectively differentiable) on \( \mathbb{R}_+ \) if for every finite subinterval of \( \mathbb{R}_+ \) there is a finite number of points where \( f \) is not continuous (respectively differentiable). This assumption could be relaxed, but it allows a suitable generality of our main result.

18 Note that, due to the term \( C_v, \) this model is a delayed-time optimal control model, which can be solved using a suitable version of the Pontryagin maximum principle (Kamien & Schwartz, 2012)

19 The value of \( C_0 \) is determined separately for each model version to simplify the result. The constraint \( C(0) = C_0 \) is of more technical nature related to the delayed term \( C_v. \) Note that this constraint only applies to the initial time \( t = 0 \) after which \( C \) is allowed to take any non-negative value. This constraint thus only affects the very beginning of the growth path.
\[ \pi'(K^*) = (\rho + \delta) \frac{1 - hA}{1 - h}. \]

Let \( B^* \) be the solution to \( \dot{B} = \delta K^* - \alpha B \) and \( B(0) = K(0) \), i.e.,

\[ B^*(t) = \frac{K^*(\alpha - \delta)e^{-\alpha t} + \delta}{\alpha}. \]

Assume \( P = (D, C, 0, K_0) \) such that \( C = m(\pi(K^*) - aB^*) \), \( T = \tau[(1 - m)(\pi(K^*) - aB^*) + zm(\pi(K^*) - aB^*)] \) and \( K = K^* \). Then \( B = B^* \) and \( P = (D, C, 0, K_0) \) ∈ \( \Gamma \). Let \( C_0 = C(0) \).

We show that \( P = (D, C, 0, K_0) \) is an optimal policy.

Define the constants \( \lambda_1 \) and \( \lambda_2 \) as

\[ \lambda_1 = 1 - hA \quad \text{and} \quad \lambda_2 = hA. \]

Then \( \lambda_1 + \lambda_2 = 1 \) and \( 0 < \lambda_1, \lambda_2 < 1 \). Let \( P' = (D', C', 0, K_0') \) be any other admissible solution. Consider the differences \( \Delta D = D - D' \), etc. Since \( P' \) is chosen arbitrarily, the proof of Case 2 is completed once we show that

\[ J := \int_0^\infty \Delta D \, dt - \Delta K_0 \geq 0. \]

It follows from the constraints of the problem that

\[ \Delta D = (1 - \tau)\Delta \pi(K) + \taua\Delta B + \tau(\Delta C - \Delta C_0) - \lambda_1(\Delta K + \delta\Delta K) - \lambda_2(\Delta B + \alpha\Delta B). \]

We apply the following three facts: First, by the concavity of \( \pi \) and the identity \( K = K^* \),

\[ (1 - h)\Delta \pi(K) \geq (1 - h)\pi'(K)\Delta K = (\rho + \delta)\lambda_1\Delta K. \]

Second, since \( \Delta C_0(t) = \Delta C_0 = 0 \), for \( t < \nu \), we get by a change of variables,

\[ \int_0^\infty \Delta C_\nu E \, dt = \int_0^\infty \Delta C_\nu E \, dt = \int_0^\infty \Delta C(t - \nu)E(t) \, dt = \int_0^\infty \Delta C(t)E(t + \nu) \, dt = c \int_0^\infty \Delta C E \, dt. \quad (A.x) \]

Third, by the definition of \( \lambda_2 \),

\[ h\alpha = (\rho + \alpha)\lambda_2. \]

Inserting these facts to \( J \) and using the upper estimate \( C \leq m(\pi(K) - aB) \) yields us the lower estimate
\[ J + \Delta K_0 = \int_0^\infty \left( (1 - \tau)\Delta \pi(K) + \tau \alpha \Delta B + \tau (\Delta C - z \Delta C_v) - \lambda_1 (\Delta \dot{K} + \delta \Delta K) - \lambda_2 (\Delta \dot{B} + \alpha \Delta B) \right) E \, dt \]
\[ = \int_0^\infty \left( (1 - \tau)\Delta \pi(K) + \tau \alpha \Delta B + \tau (1 - cz) \Delta C - \lambda_1 (\Delta \dot{K} + \delta \Delta K) - \lambda_2 (\Delta \dot{B} + \alpha \Delta B) \right) E \, dt \]
\[ \geq \int_0^\infty \left( (1 - \tau)\Delta \pi(K) + \tau \alpha \Delta B + \tau (1 - cz) m(\Delta \pi(K) - \alpha \Delta B) - \lambda_1 (\Delta \dot{K} + \delta \Delta K) - \lambda_2 (\Delta \dot{B} + \alpha \Delta B) \right) E \, dt \]
\[ = \int_0^\infty \left( (\rho + \delta) \lambda_1 \Delta K + (\rho + \alpha) \lambda_2 \Delta B - \lambda_1 (\Delta \dot{K} + \delta \Delta K) - \lambda_2 (\Delta \dot{B} + \alpha \Delta B) \right) E \, dt \]
\[ = \int_0^\infty \left( -\lambda_1 (\Delta \dot{K} - \rho \Delta K) - \lambda_2 (\Delta \dot{B} + \rho \Delta B) \right) E \, dt = \lambda_1 \Delta K(0) + \lambda_2 \Delta B(0) = \Delta K_0. \]

This completes the proof of Case 2.

We next consider Case 3. Assume that \( \frac{ah}{\rho + \alpha} \geq ctz. \) Define the number \( K^* > 0 \) by
\[ \pi'(K^*) = (\rho + \delta) \frac{1 - h - \alpha}{\alpha + \rho} \]
Let \( B^* \) be the solution to \( \dot{B} = \delta K^* - \alpha B \) and \( B(0) = K(0), \) i.e.,
\[ B^*(t) = \frac{K^* (\alpha - \delta) e^{-\alpha t} + \delta}{\alpha} \]
Assume \( P = (D, C, G, K_0) \) such that \( C = m(\pi(K^*) - \alpha B^*), \) \( G = 0, \) and \( K_0 = K^*. \)
Define the constants \( \lambda_1 \) and \( \lambda_2 \) as
\[ \lambda_1 = 1 - \frac{ah}{\alpha + \rho} \quad \text{and} \quad \lambda_2 = \frac{ah}{\alpha + \rho} \]
Then \( \lambda_1 + \lambda_2 = 1 \) and \( 0 < \lambda_1, \lambda_2 < 1. \) Let \( P' = (D', C', G', K_0') \) be any other admissible solution. Consider the differences \( \Delta D = D - D', \) etc. Since \( P' \) is chosen arbitrarily, the proof of Case 3 is completed once we show that
\[ J := \int_0^\infty \Delta DE \, dt - \Delta K_0 \geq 0. \]
It follows from the constraints of the problem that
\[ \Delta D = (1 - \tau)\Delta \pi(K) + \tau \alpha \Delta B + \tau (\Delta C - z (\Delta C_v - \Delta G_v)) - \lambda_1 (\Delta \dot{K} + \delta \Delta K) - \lambda_2 (\Delta \dot{B} + \alpha \Delta B) - \lambda_2 \Delta G. \]
Using the change of variables \((A.x),\) we get
\[ J + \Delta K_0 = \int_0^\infty \left( (1 - \tau)\Delta \pi(K) + \tau a \Delta B + (1 - cz) \Delta C + (ctz - \lambda_2) \Delta G - \lambda_1 (\Delta \dot{K} + \delta \Delta K) \right. \\
\left. - \lambda_2 (\Delta \dot{B} + a \Delta B) \right) E \, dt. \]

Since \( cz \leq \frac{\lambda_2}{\tau} \leq 1 \), and using the concavity of \( \pi \) to get \( (1 - h)\Delta \pi(K) \geq (1 - h)\pi'(K)\Delta K = (\rho + \delta)\lambda_1 \Delta K \),

\[ J + \Delta K_0 \geq \int_0^\infty \left( (1 - \tau)\Delta \pi(K) + \tau a \Delta B + (1 - cz)m(\Delta \pi(K) - a \Delta B) - \lambda_1 (\Delta \dot{K} + \delta \Delta K) \right. \\
\left. - \lambda_2 (\Delta \dot{B} + a \Delta B) \right) E \, dt \\
= \int_0^\infty \left( (1 - h)\Delta \pi(K) + ha \Delta B - \lambda_1 (\Delta \dot{K} + \delta \Delta K) - \lambda_2 (\Delta \dot{B} + a \Delta B) \right) E \, dt \\
\geq \int_0^\infty \left( (\rho + \delta)\lambda_1 \Delta K + (\alpha + \rho)\lambda_2 \Delta B - \lambda_1 (\Delta \dot{K} + \delta \Delta K) - \lambda_2 (\Delta \dot{B} + a \Delta B) \right) E \, dt = \Delta K_0 \]

We next consider Case 4. Assume \( m \) is “small” [exact value to be determined in later analysis] and \( A' = rA/\tau \leq cz \). Let \( r = \tau(1 - m) + \tau A'm \). Define the number \( K^* > 0 \) by

\[ \pi'(K^*) = (\rho + \delta) \frac{1 - rA}{1 - r}. \]

Let \( B^* \) be the solution to the differential equation \( \dot{B} = \delta K^* - m\pi(K^*) - a(1 - m)B \) and \( B(0) = K^* \). [<- helppo ratkaista] \n
Assume \( P = (D, C, G, K_0) \) such that \( C = m(\pi(K^*) - aB^*) \), \( G = m(\pi(K^*) - aB^*) \), \( T = \tau(1 - m)(\pi(K^*) - aB^*) \) and \( K = K^* \). Then \( B = B^* \) and \( P = (D, C, G, K_0) \in \Gamma \). Let \( C_0 = C(0) \).

Define the constants \( \lambda_1 \) and \( \lambda_2 \) as

\[ \lambda_1 = 1 - \frac{\tau a(1 - m)}{a(1 - m) + \rho} \quad \text{and} \quad \lambda_2 = \frac{\tau a(1 - m)}{a(1 - m) + \rho}. \]

Then \( \lambda_1 + \lambda_2 = 1 \) and \( 0 < \lambda_1, \lambda_2 < 1 \). Let \( P' = (D', C', G', K_0') \) be any other admissible solution. Consider the differences \( \Delta D = D - D' \), etc. Since \( P' \) is chosen arbitrarily, the proof of Case 4 is completed once we show that

\[ J : = \int_0^\infty \Delta DE \, dt - \Delta K_0 \geq 0. \]

It follows from the constraints of the problem that

\[ \Delta D = (1 - \tau)\Delta \pi(K) + \tau a \Delta B + \tau (\Delta C - z(\Delta C_v - \Delta G_v)) - \lambda_2 \Delta G - \lambda_1 (\Delta \dot{K} + \delta \Delta K) - \lambda_2 (\Delta \dot{B} + a \Delta B). \]

Using the change of variables \( (A.x) \), we get

\[ J + \Delta K_0 = \int_0^\infty \left( (1 - \tau)\Delta \pi(K) + \tau a \Delta B + (\tau - \lambda_2 c) \Delta C + (ctz - \lambda_2)(\Delta G - \Delta C) - \lambda_1 (\Delta \dot{K} + \delta \Delta K) \right. \\
\left. - \lambda_2 (\Delta \dot{B} + a \Delta B) \right) E \, dt. \]
Since $\tau - \lambda_2 c \geq 0$ and $ctz - \lambda_2 \geq 0$, we get

$$J + \Delta K_0 \geq \int_0^\infty \left( (1 - \tau)\Delta \pi(K) + \tau a \Delta B + (\tau - \lambda_2 c)m(\Delta \pi(K) - a \Delta B) - \lambda_1(\Delta \dot{K} + \delta \Delta K) - \lambda_2(\Delta \dot{B} + a \Delta B) \right) E \, dt$$

Using the concavity of $\pi$ and the identity $K = K^*$, we get, as before,

$$(1 - r)\Delta \pi(K) \geq (1 - r)\pi'(K)\Delta K = (\rho + \delta)\lambda_1 \Delta K,$$

and $ra = (\rho + \alpha)\lambda_2$, so that

$$J + \Delta K_0 \geq \int_0^\infty \left( (\rho + \delta)\lambda_1 \Delta K + (\alpha + \rho)\lambda_2 \Delta B - \lambda_1(\Delta \dot{K} + \delta \Delta K) - \lambda_2(\Delta \dot{B} + a \Delta B) \right) E \, dt = \Delta K_0$$

We next sketch the proof of Case 5. Assume $m$ is “large” [exact value to be determined in later analysis; hypothesis: $m(\pi(K^*) - aB^*(t)) \geq \delta K^*$ for some $t$]. Assume that $A' = \frac{ra}{\tau} < cz < 1$. Let

$$\pi'(K^*) = (\rho + \delta) \frac{1 - ctz}{1 - h}.$$ 

Let $B'(t) = K^* e^{-at}$. Assume $P = (D, C, \gamma, K_0)$ such that $K = K^*$, $C = m(\pi(K^*) - aB^*)$, $T = \tau [1 - m(\pi(K^*) - aB^*)] + z(m(\pi(K^*) - aB^*) - \delta K^*)$ and $G = \delta K^*$. Then $B = B^*$, $I = \delta K^*$ and $P = (D, C, \gamma, K_0) \in \Gamma$. $C_0 = C(0)$.

Define the constants $\lambda_1$ and $\lambda_2$ as

$$\lambda_1 = 1 - ctz, \quad \lambda_2 = \frac{ha}{\alpha + \rho} \quad \text{and} \quad q = ctz - \frac{ha}{\alpha + \rho}.$$ 

Then $\lambda_1 + \lambda_2 + q = 1$ and $0 < \lambda_1, \lambda_2, q < 1$. Let $P' = (D', C', K_0')$ be any other admissible solution. As before, consider the differences $\Delta D = D - D'$, etc. Since $P'$ is chosen arbitrarily, the proof of Case 5 is completed once we show that

$$J := \int_0^\infty \Delta DE \, dt - \Delta K_0 \geq 0.$$ 

It follows from the constraints of the problem that

$$\Delta D = (1 - \tau)\Delta \pi(K) + \tau a \Delta B + \tau (\Delta C - z(\Delta C - \Delta G_v)) - \lambda_1(\Delta \dot{K} + \delta \Delta K) - \lambda_2(\Delta \dot{B} + a \Delta B) - q \Delta I - \lambda_2 \Delta G.$$
Using the change of variables \((A.x)\) [with \(C_v = G_v\) for \(t < v!\)] and \(1 - cz \geq 0\) as well as \(q \geq 0\), we get

\[
J + \Delta K_0 = \int_0^\infty \left( (1 - \tau)\Delta \pi(K) + \tau a \Delta B + \tau(1 - cz)\Delta C + q(\Delta G - \Delta I) - \lambda_1(\Delta \hat{K} + \delta \Delta K) \right.
\]

\[
- \lambda_2(\Delta \hat{B} + a \Delta B) \big) E \, dt
\]

\[
\geq \int_0^\infty \left( (1 - h)\Delta \pi(K) + h a \Delta B - \lambda_1(\Delta \hat{K} + \delta \Delta K) - \lambda_2(\Delta \hat{B} + a \Delta B) \big) E \, dt.
\]

Using the concavity of \(\pi\) and the identity \(K = K^*\), we get, as before,

\[
(1 - h)\Delta \pi(K) \geq (1 - h)\pi'(K)\Delta K = (\rho + \delta)\lambda_1 \Delta K,
\]

and \(ha = (\rho + a)\lambda_2\), so that

\[
J + \Delta K_0 \geq \int_0^\infty \left( (\rho + \delta)\lambda_1 \Delta K + (\alpha + \rho)\lambda_2 \Delta B - \lambda_1(\Delta \hat{K} + \delta \Delta K) - \lambda_2(\Delta \hat{B} + a \Delta B) \big) E \, dt
\]

\[
= \Delta K_0(\lambda_1(0) + \lambda_2(0))
\]

NB! Work in progress: The RHS is not quite the same as \(\Delta K_0 \to\) There exists a growth regime yet to be identified.

We next turn to the proof of Case 6. Assume \(m\) is “large” [exact value to be determined in later analysis; hypothesis: \(m(\pi(K^*) - aB'(t)) \geq \delta K^*\) for some \(t\)]. Assume that \(cz > 1\).

Define the functions \(K, B, \lambda_1, \lambda_2\) and numbers \(K^* > 0, t^* > 0\) by the equations

\[
\dot{K} = m(\pi(K) - aB) - \delta K \quad \text{for} \quad 0 < t < t^*
\]

\[
\dot{\lambda}_1 = (\rho + \delta - mn'(K))\lambda_1 - (1 - \tau)(1 - m)n'(K) \quad \text{for} \quad 0 < t < t^*
\]

\[
\dot{\lambda}_2 = (\rho + \alpha)\lambda_2 - \tau \alpha + (\lambda_1 - 1 + \tau)m \alpha \quad \text{for} \quad 0 < t < t^*
\]

\[
K(t) = K^* \quad \text{for} \quad t > t^*
\]

\[
\lambda_1(t) = 1 - \tau \quad \text{for} \quad t > t^*
\]

\[
\lambda_2(t) = \frac{\tau \alpha}{\alpha + \rho} \quad \text{for} \quad t > t^*
\]

\[
B(t) = K(0)e^{-\alpha t}
\]

\[
\pi'(K^*) = \rho + \delta
\]

\[
\lambda_1(0) + \lambda_2(0) = 1,
\]

so that \(K, \lambda_1, \lambda_2\) are continuous.

Assume \(P = (D, C, G, K_0)\) such that \(C = G = I, I = m(\pi(K) - aB)\) for \(0 < t < t^*\) and \(I = \delta K^*\) for \(t > t^*\). Then \(T = \tau(1 - m)[\pi(K) - aB]\) for \(0 < t < t^*\), \(T = \tau[\pi(K^*) - aB^* - \delta K^*]\) for \(t > t^*\) and \(P = (D, C, G, K_0) \in \Gamma\). Let \(C_0 = C(0)\).

We show that \(P = (D, C, G, K_0)\) is an optimal policy.
Let \( q = 1 - \lambda_1 - \lambda_2 \). Let \( p = \lambda_1 - (1 - \tau) \). We have \( \lambda_1 + \lambda_2 + q = 1 \) and \( 0 < \lambda_1, \lambda_2, q < 1 \). Let \( P' = (D', C', K'_0) \) be any other admissible solution. As before, consider the differences \( \Delta D = D - D' \), etc. Since \( P' \) is chosen arbitrarily, the proof of Case 6 is completed once we show that

\[
J \equiv \int_0^\infty \Delta DE \, dt - \Delta K_0 \geq 0.
\]

It follows from the constraints of the problem that

\[
\Delta D = (1 - \tau)\Delta \pi(K) + \tau a \Delta B + \tau(\Delta C - z(\Delta c_v - \Delta G_v)) - \lambda_1(\Delta \dot{K} + \delta \Delta K) - \lambda_2(\Delta \dot{B} + a \Delta B) - q \Delta I - \lambda_2 \Delta G
\]

\[
= p \Delta c_v + (1 - \tau)\Delta \pi(K) + \tau a \Delta B - (\lambda_1 - 1)(\Delta c - \Delta G) - \tau z(\Delta c_v - \Delta G_v)
\]

\[
- \lambda_1(\Delta \dot{K} + \delta \Delta K) - \lambda_2(\Delta \dot{B} + a \Delta B) - q(\Delta I - \Delta G).
\]

Since \( C = m(\pi(K) - aB) \), we have

\[
\Delta D \geq pm \Delta(\pi(K) - aB) + (1 - \tau)\Delta \pi(K) + \tau a \Delta B - (\lambda_1 - 1)(\Delta c - \Delta G) - \tau z(\Delta c_v - \Delta G_v)
\]

\[
- \lambda_1(\Delta \dot{K} + \delta \Delta K) - \lambda_2(\Delta \dot{B} + a \Delta B)
\]

\[
= (pm + 1 - \tau)\Delta \pi(K) - (pm - \tau) a \Delta B - (\lambda_1 - 1)(\Delta c - \Delta G) - \tau z(\Delta c_v - \Delta G_v)
\]

\[
- \lambda_1(\Delta \dot{K} + \delta \Delta K) - \lambda_2(\Delta \dot{B} + a \Delta B)
\]

Again, by the concavity of \( \pi \),

\[
(pm + 1 - \tau)\Delta \pi(K) \geq (pm + 1 - \tau)\pi'(K) \Delta K = \left((\rho + \delta)\lambda_1 - \dot{\lambda}_1\right) \Delta K,
\]

so that

\[
\Delta D \geq \left((\rho + \delta)\lambda_1 - \dot{\lambda}_1\right) \Delta K + \left((\rho + \alpha)\lambda_2 - \dot{\lambda}_2\right) \Delta B - (\lambda_1 - 1)(\Delta c - \Delta G) - \tau z(\Delta c_v - \Delta G_v)
\]

\[
- \lambda_1(\Delta \dot{K} + \delta \Delta K) - \lambda_2(\Delta \dot{B} + a \Delta B)
\]

\[
= -(\lambda_1 - 1)(\Delta c - \Delta G) - \tau z(\Delta c_v - \Delta G_v) - \lambda_1(\Delta \dot{K} - \rho \Delta K) - \dot{\lambda}_1 \Delta K - \lambda_2(\Delta \dot{B} - \rho \Delta B)
\]

\[
- \dot{\lambda}_2 \Delta B
\]

Using the change of variables (A.x) and the assumption \( cz > 1 \), yields us now the lower estimate

\[
\int_0^\infty \Delta DE \, dt \geq \int_0^\infty \left(-\lambda_1 - 1 + \tau cz(\Delta c - \Delta G) - \lambda_1(\Delta \dot{K} - \rho \Delta K) - \dot{\lambda}_1 \Delta K - \lambda_2(\Delta \dot{B} - \rho \Delta B) - \dot{\lambda}_2 \Delta B\right) \, dt
\]

\[
\geq \int_0^\infty -\lambda_1(\Delta \dot{K} - \rho \Delta K) - \dot{\lambda}_1 \Delta K - \lambda_2(\Delta \dot{B} - \rho \Delta B) - \dot{\lambda}_2 \Delta B \, dt
\]

\[
= \lambda_1(0) \Delta K(0) + \lambda_2(0) \Delta B(0) = \Delta K_0.
\]