Tax Havens and Welfare in Non-Haven Countries in a Dynamic World

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Abstract

This paper analyzes the short and long term effects that tax havens have on the welfare in non-haven countries when multinational enterprises (MNEs) use internal debt to minimize their tax liability. This profit shifting channel lowers the tax rate sensitivity of investment in the non-haven countries and may impact positively the equilibrium tax rates and welfare. In a static setting, the short and long term effects of profit shifting on welfare are identical and unambiguously positive. However, in a dynamic model, profit shifting lowers welfare in the short term and increases it in the long term, if the redistribution motive of the government is not too strong. While the long term gain overcompensates the initial welfare decline, the nonmonotonic welfare effects have important implications both for empirical research and policy.

Keywords: tax havens, profit shifting, internal debt
JEL Code: F23; H25; H7;

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1 Introduction

Multinational enterprises (MNEs) take advantage of the existence of tax havens to shift profits away from high-tax countries. This behavior erodes the tax revenues in the latter countries and has led the OECD to start a base erosion and profit shifting (BEPS) initiative (OECD, 2013a,b). The aim of the BEPS initiative is to develop measures to address the MNEs’ aggressive tax planning strategies.

However, there is no consensus in the economic literature on how profit shifting to tax havens affects high-tax countries’ welfare. On the one hand, elimination of tax havens is good for non-haven countries if it improves public good provision in the latter countries (Slemrod and Wilson, 2009; Hauffer and Runkel, 2012) or if it removes the secrecy of firm ownership (Weichenrieder and Xu, 2019). However, elimination of tax havens may have an ambiguous impact on non-haven jurisdictions’ welfare if it intensifies the tax competition among the high-tax countries (Johannesen, 2010) or if it is only partial and lowers competition among the remaining havens (Elsayyad and Konrad, 2012). International tax planning may be good for non-haven welfare if MNEs’ organizational form responds to tax discrimination (Bucovetsky and Hauffer, 2008), if governments respond to tax planning by changing their tax enforcement strategies (Chu, 2014), or in the presence of lobbying by the owners of immobile capital (Chu et al., 2015).

The most prominent work deriving an unambiguously positive welfare impact of international tax planning is by Hong and Smart (2010). In their model, the equilibrium tax rate is increasing in the amount of profits shifted to tax havens via internal debt. Thus, despite the negative tax base effect of profit shifting on tax revenues, the welfare in the non-haven country is monotonically increasing in the amount of internal debt.

A critical feature of the Hong and Smart (2010) framework is that it is static. Thus, a change in profit shifting results in an immediate jump to a new equilibrium. However, capital is a stock that can adjust only gradually to a change in its user cost.

I build a dynamic version of the Hong and Smart (2010) model to reassess the welfare implications of profit shifting when capital adjustment is costly. The economy consists of one high-tax (non-haven or host) country which hosts a national firm and a subsidiary of a foreign-owned MNE. Workers supply labor that is perfectly mobile between the national and multinational sector but immobile internationally. The MNE invests mobile capital in the host country and capital adjustment is subject to adjustment costs. These costs give rise to transitional dynamics. The model of Hong and Smart (2010) is a special case of my model when the capital adjustment costs are zero. The host country government taxes profits at a constant statutory tax rate and has incentives to redistribute income from the owners of national firms to the workers.

I derive three main results. First, the relation between profit shifting and the MNE’s
steady state capital stock is ambiguous. If the model is static, the government responds to an increase in profit shifting by strongly increasing its tax rate such that the user cost of capital rises and the capital stock declines. However, if the economy is not static and the redistribution motive of the government is not too strong, the government’s response to more profit shifting is less pronounced, the user cost of capital declines and capital becomes an increasing function of internal debt.

Second, the short term and long term welfare effects of an increase in profit shifting are, in general, ambiguous. If the redistribution motive of the government is sufficiently weak, welfare declines unambiguously in the short term and increases in the long term. The intuition is that, in this case, the government responds to more profit shifting by only weakly increasing the tax rate. Thus, the direct negative impact of profit shifting on tax revenues lowers welfare in the short-run by more than the increase in the tax rate raises it. In the long-run, the capital stock adjusts to a higher steady state and raises welfare.

The result of a nonmonotonous welfare impact of profit shifting has two important implications. First, it means that the timing of the empirical evaluation of policy reforms may be crucial. A policy that restricts tax planning may have positive short-term and negative long-term effects and vice versa. Moreover, the implementation of reforms that are beneficial in the long term may include short term costs. Hence, these costs may need to be identified and addressed by such reforms ex-ante.

Third, the discounted sum of welfare changes is unambiguously positive. This result reinforces the prediction by Hong and Smart (2010). However, it hides the possible nonmonotonic evolution of welfare along the transition path.

In Section 5, the model is extended to consider a time-varying statutory tax rate. The results of the main model remain qualitatively unchanged.

This paper is related to a large literature on the implications of profit shifting to tax havens for non-haven countries’ welfare. The analysis of Hong and Smart (2010) has also been extended by Gresik et al. (2015). Gresik et al. (2015) consider additionally that MNEs shift profits through transfer price manipulation. They find that a less developed country, that cannot effectively monitor the transfer pricing decisions of multinational firms, may lose welfare by allowing profit shifting to tax havens.

Desai et al. (2006a) find empirical evidence that high sales and investment growth firms are more likely to operate in tax havens. In a theoretical model, Desai et al. (2006b) explain this empirical result through a theoretical model. The use of tax havens impacts positively the economic activity in non-haven countries, as it raises the return from investment in high-tax jurisdictions.

Peralta et al. (2006) examine whether profit shifting between non-haven countries has beneficial welfare effects when the MNE also makes a choice for the location of its productive subsidiary. They find that some high-tax countries may have incentives not
to monitor profit shifting in order to become an attractive location for production.

This paper differs from the remaining literature by being the first to develop a
dynamic model which differentiates between the short term and long term effects of
profit shifting on non-haven countries’ welfare. I find that profit shifting via internal
debt may exert non-monotone welfare effects over time. Hence, even if a reform that
relaxes the thin-capitalization rules to stimulate the use of internal debt is overall welfare
improving, additional measures may be needed in the short term, if it initially lowers
welfare.

The rest of the paper is structured as follows. Section 2 presents the model. I derive
the optimal tax policy in Section 3 and the welfare effects of profit shifting in Section
4. Section 5 extends the model to a time-varying tax rate, while Section 6 concludes.

2 The Model

Consider a small open high-tax economy. There are two types of infinitely-lived agents
in the economy: workers and entrepreneurs. The economy produces a single homoge-
neous good in two sectors: domestic firms owned by entrepreneurs and a subsidiary of
an MNE whose parent company is foreign-based. The representative worker supplies
one unit of labor which is fully mobile between the national and multinational sectors.

The domestic sector produces the homogeneous good using a constant returns to
scale technology \( G(D, L^d) \), where \( D \) is a fixed immobile capital stock, \( L^d \) denotes the
labor input and \( G(\cdot) \) has positive, but diminishing marginal products. Denote the time-
invariant statutory tax rate as \( \tau \) (the assumption of a time-invariant tax will be relaxed
later), the period \( t \) labor employed by the national sector as \( L^d_t \) and the period \( t \) wage
rate as \( w_t \). Then, the after-tax profit of the entrepreneurial firm in period \( t \) is

\[
\pi^D_t = (1 - \tau)(G(D, L^d_t) - w_t L^d_t).
\]  

(1)

In each period, the entrepreneurs maximize the after-tax profit (1) over the labor input
\( L^d_t \), which results in the labor demand equation:

\[
G_L(D, L^d_t) = w_t.
\]  

(2)

I model a dynamic version of the MNE’s subsidiary considered by Hong and Smart
(2010) by following Turnovsky and Bianconi (1992) and Wildasin (2003). The multi-
national firm uses the constant returns technology \( F(K, L^m) \), where \( K \) is the capital
stock, \( L^m \) the labor input and \( F(\cdot) \) has positive, but diminishing marginal products.
The firm has an initial capital stock \( K(0) = K_0 \). Capital is fully equity financed, either
through new equity issues or retained earnings.
The MNE has a financial center in a tax haven country with a zero corporate tax rate. It can lower the tax liability of its productive subsidiary by channeling a part of the equity financing through the financial center, which in turn provides internal debt to the subsidiary at an exogenous world interest rate \( r \). Following Hong and Smart (2010); Haufler and Runkel (2012), there are no deadweight costs of using internal debt. Without loss of generality, internal debt is constrained by the government to not exceed an exogenous proportion \( b \in [0,1] \) of the capital stock. In the absence of deadweight costs, the firm would like to use as much internal debt as possible due to its tax advantage. Hence, the amount of internal debt in period \( t \) is \( bK_t \) and the interest costs amount to \( rbK_t \). These interest costs also equal the net profit generated by the financial center. Define the net profit of the MNE’s subsidiary in period \( t \) as its gross profit, \( F(K_t, L^m_t) - w_tL^m_t \), minus the interest costs and tax payments. Then, the sum of the subsidiary’s and financial center’s net profits in period \( t \) equal

\[
\tilde{\pi}^M_t = F(K_t, L^m_t) - w_tL^m_t - rbK_t - \tau[F(K_t, L^m_t) - w_tL^m_t - rbK_t] + rbK_t.
\] (3)

The profit \( \tilde{\pi}^M_t \) can either be used to pay dividends \( D_t \) or held as retained earnings \( RE_t \) to finance new investment.

The MNE augments the capital stock in period \( t \), \( K_t \), at the rate \( I_t \) such that the amount of investment is \( I_tK_t \). Assuming, without loss of generality, that capital does not depreciate, the capital stock evolves according to\(^{12}\)

\[
\dot{K}_t = I_tK_t.
\] (4)

Moreover, the firm incurs convex adjustment costs \( C(I_t)K_t \), where \( \text{sgn}\{C'\} = \text{sgn}\{I\}, C'' > 0 \) and \( C(0) = C'(0) = 0 \). Therefore, the costs of capital adjustment in period \( t \) are \( (I_t + C(I_t))K_t \). These costs are financed through retained earnings \( RE_t \) and new equity issues \( q_tE_t \) by the parent company, where \( q_t \) is the price of equity and \( E_t \) denotes the stock of existing equity in period \( t \).

Subtraction of the capital adjustment costs from the net profit \( \tilde{\pi}^M_t \) gives the net cash-flow generated by the MNE’s subsidiary in period \( t \):

\[
\pi^M_t = F(K_t, L^m_t) - w_tL^m_t - (I_t + C(I_t))K_t - \tau[F(K_t, L^m_t) - w_tL^m_t - rbK_t].
\] (5)

Denote the value of equity in period \( t \) as \( V_t = q_tE_t \). The objective of the firm is to choose the optimal paths \( I_t \) and \( L^m_t \) to maximize \( V_0 \), which is equivalent to maximizing

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\(^{1}\)We assume zero depreciation to remain as close as possible to the static model of Hong and Smart (2010). The inclusion of these costs does not affect the results.

\(^{2}\)A dot denotes a time derivative.
Thus, the value of the subsidiary in period 0 is the present value of its future net cash-flow, discounted at the interest rate $r$. Equations (5) and (6) are identical to the equations for net cash-flow and firm value in the dynamic tax competition model of Wildasin (Wildasin, 2003, 2011) when one sets the internal debt ratio $b$ equal to zero.

In Appendix B, I show that the MNE’s optimal paths of $I_t$ and $L_t^m$ satisfy the following equations:

$$\dot{I}_t = \frac{1}{C''(I_t)} \left[ r(1 - b\tau) + C(I_t) + C'(I_t)(r - I_t) - F_K(K_t, L_t^m)(1 - \tau) \right], \quad (7)$$

$$w_t = F_L(K_t, L_t^m). \quad (8)$$

Equation (8) equates the marginal product of labor to its marginal cost, while (7) determines the optimal change in investment over time, $\dot{I}_t$. The right-hand side of (7) gives the difference between the costs of new investment and its marginal product. Note that in steady state capital is constant, $\dot{K} = 0$, and, according to (4), $I = 0$. Hence, if a shock increases the marginal costs of investment, the right-hand side of (7) becomes positive. In this situation, the firm disinvests ($I < 0$) and to reach steady state, investment must increase to zero, i.e., $\dot{I} > 0$.

Finally, the labor market must clear in each period $t$. Thus, we require

$$L_t^d + L_t^m = 1. \quad (9)$$

To see the similarity to the model of Hong and Smart (2010), derive the steady state where $\dot{I} = \dot{K} = 0$. Denote steady state variables with a $\tilde{}$. It is characterized by

$$F_K(\tilde{K}, \tilde{L}^m) = \frac{r(1 - b\tau)}{1 - \tau}, \quad (10a)$$

$$F_L(\tilde{K}, \tilde{L}^m) = G_L(D, \tilde{L}^d) = \tilde{w}, \quad (10b)$$

$$\tilde{L}^m + \tilde{L}^d = 1, \quad (10c)$$

$$\tilde{I} = 0. \quad (10d)$$

This steady state is identical to the static capital and labor market equilibria of Hong and Smart (2010). In the case of zero adjustment costs, the MNE can change immediately its capital stock and the model is always in steady state. Therefore, the static model is a special case of the model developed here when $C(I) = 0$. 

(see Appendix A for a derivation)
The labor demand equations (2) and (8) together with the labor market clearing condition (9) define the labor inputs \( L^d_t, L^m_t \) as well as the wage rate \( w_t \) as implicit functions of the capital stock \( K_t \). Denote these functions as \( L^m_t \equiv \ell^m_t(K_t), L^d_t \equiv \ell^d_t(K_t), w_t \equiv \omega_t(K_t) \). A total differential Equations (2), (8) and (9) with respect to \( L^m_t, L^d_t, w_t \) and \( K_t \) gives
\[
\frac{d\ell^m_t}{dK_t} = -\frac{F_{LK}}{F_{LL} + G_{LL}}, \quad \frac{d\ell^d_t}{dK_t} = -\frac{d\ell^m_t}{dK_t}, \quad \frac{d\omega_t}{dK_t} = \frac{G_{LL}F_{LK}}{F_{LL} + G_{LL}}.
\] (11)

To interpret (11), suppose that capital and labor are complements in production, i.e., \( F_{LK} > 0 \). Then, an increase in the capital stock makes labor more productive, which raises the demand for labor in the international sector \( (d\ell^m_t/dK_t > 0) \). The wage rate must increase to balance the labor market \( (d\omega_t/dK_t > 0) \), which lowers the demand for labor in the national sector.

Next, I derive the comparative dynamic effects of a change in the tax rate in period 0 on the capital stock in periods \( t \), where \( t \geq 0 \). Following Wildasin (2003, 2011), Appendix C proves the following:

**Lemma 1.** Suppose the government changes the tax rate \( \tau \) in period 0 and keeps it constant for all future periods. Then, the change in the capital stock in periods \( t \geq 0 \) is
\[
\frac{dK_t}{d\tau} = \frac{d\tilde{K}}{d\tau} \left[ 1 - e^{\mu_1 t} \right],
\] (12)

where \( \mu_1 \) is the speed of convergence to the steady state and is determined by
\[
\mu_1 = \frac{r - \sqrt{r^2 - 4(1-\tau)F_{KK}G_{LL}K}}{2C''(F_{LL} + G_{LL})} < 0,
\] (13)

while \( d\tilde{K}/d\tau \) is the change in the steady state capital stock, given by
\[
\frac{d\tilde{K}}{d\tau} = \frac{(F_K(\tilde{K}, \tilde{L}^m) - rb)(F_{LL} + G_{LL})}{(1-\tau)F_{KK}G_{LL}} < 0.
\] (14)

**Proof:** See Appendix C.

According to Equation (12), in the period of the tax change \( (t = 0) \), the capital stock remains unchanged: \( dK_0 = 0 \). The reason is that capital is a stock that cannot adjust immediately. When \( t \) becomes large, the exponential term in (12) vanishes and the change in the capital stock approached the negative long-term effect \( d\tilde{K}/d\tau < 0 \). The speed of convergence is \( \mu_1 \). If there are not capital adjustment costs and, thus, \( C'' = 0 \), then \( \mu_1 \to -\infty \) and adjustment is instantaneous. This is the special case of a static model. The higher the change in the marginal adjustment costs \( C'' \) is, the
slower is the rate of adjustment $\mu_1$. Lastly, the comparative dynamic effects on the labor inputs and the wage rate in periods $t \geq 0$ follow from Equations (11) and (12).

3 The Government

Following Hong and Smart (2010), I assume that the government’s objective is to re-distribute income from entrepreneurs to workers. It transfers the tax revenues in a lump-sum way to the workers. The workers do not save and their consumption equals the total income in period $t$:

$$C^W_t = w_t + T_t,$$

(15)

where $T_t = \tau(G(D, L^d_t) - w_t L^d_t) + \tau(F(K_t, L^m_t) - w_t L^m_t - rb K_t)$ denotes the tax revenues. The entrepreneurs also do not save and their consumption $C^E_t$ is given by

$$C^E_t = \pi^D_t.$$

(16)

The government maximizes the welfare function $\Omega_t = C^W_t + \beta C^E_t$ for $\beta \in [0, 1]$, where $\beta$ strictly less than one represents preferences for redistribution of income toward workers. Suppose that the government uses the same discount rate $r$ as the multinational firm. Then, it solves

$$\max_\tau \int_0^\infty \Omega_t e^{-rt} dt,$$

(17)

taking into account the impact of taxation on the capital stock $dK_t/d\tau$, as well as $L^m_t = \ell^m_t(K_t)$, $L^d_t = \ell^d_t(K_t)$ and $w_t = \omega_t(K_t)$. I derive the optimal tax rate in Appendix D. Denote this tax rate as $\tau^*$. It is determined by

$$\frac{\mu_1}{r - \mu_1} \tau^* (F_K(\tilde{K}, \tilde{L}^m) - rb) \frac{d\tilde{K}}{d\tau} = (1 - \beta) \left[ G(D, \tilde{L}^d) - \tilde{w} \tilde{L}^d - \frac{\mu_1}{r - \mu_1} (1 - \tau^*) \tilde{L}^d \frac{d\tilde{w}}{d\tau} \right]$$

$$+ \frac{r}{r - \mu_1} (F_K(\tilde{K}, \tilde{L}^m) - rb) \tilde{K},$$

(18)

where the term $d\tilde{w}/d\tau$ is defined in Equation (D.5) in Appendix D. The left-hand side of (18) gives the marginal costs of an increase in the tax rate: it drives away the mobile capital of the multinational sector. The marginal benefits are on the right-hand side of (18). The term containing $(1 - \beta)$ gives the marginal increase in welfare from additional redistribution from entrepreneurs to workers. It is positive for $\beta$ strictly less than one. The term in the second row of (18) arises due to the dynamic adjustment of the capital.
stock and is initially derived by Wildasin (2003). The slow adjustment of the capital stock following a tax rate increase creates quasi-rents during the transition period to a new steady state. Since the multinational firm is not owned by the domestic residents, the government has incentive to tax these rents and distribute them to the workers. This term is greater, the slower the adjustment rate is, i.e., the closer $\mu_1$ is to zero. In the case of an immediate adjustment, $\mu_1 \to -\infty$, there are no quasi-rents, and the term on the second row of (18) vanishes.

It is helpful for the remaining analysis to consider the special case of a very fast adjustment speed. In Appendix E, I prove the following result:

**Lemma 2.** If there are no capital adjustment costs, such that $C''(0) = 0$, then the economy is always in steady state ($\mu_1 \to -\infty$) and $\tau^*$ coincides with the optimal static tax rate from Hong and Smart (2010).

**Proof:** See Appendix E. $\square$

The intuition behind Lemma 2 is straightforward. When the adjustment to steady state is instantaneous ($\mu_1 \to -\infty$), the model collapses to the static model of Hong and Smart (2010). Therefore, in this case, the equilibrium tax rate is identical to their result.

Next, I discuss the long and short term effects of a change in the aggressiveness of profit shifting, as measured by the proportion of internal debt $b$.

## 4 Effects of Internal Debt

I start by discussing the effects of changes in the amount of internal debt, as measured by $b$, on the optimal tax rate $\tau^*$ and the long-term capital stock $\tilde{K}$. Note that the impact on $\tilde{K}$ is positive, if the user cost of capital goes down (and vice versa), where the user cost is determined by the right-hand side of Equation (10a). Hong and Smart (2010) show that an increase in $b$ raises the statutory tax rate and lowers the capital stock $\tilde{K}$, if the initial tax rate is not too high ($\tau < 1/2$). In Appendix F, I derive the following results:

**Proposition 1.** Suppose the amount of internal debt $b$ increases by $db > 0$ in period 0. If the economy is always in steady state ($\mu_1 \to -\infty$), then

$$\frac{d\tau^*}{db} > 0, \quad (19)$$

$$\frac{d\tilde{K}}{db} < 0, \quad \text{if} \quad \tau^* < \frac{1}{2}. \quad (20)$$

Suppose the economy is dynamic with $\mu_1 \in ]-\infty, 0[$. Then, there exists a value $\hat{\beta} \in [0, 1[$
such that for $\beta \in [\hat{\beta}, 1]$, the following results emerge:

$$\frac{d\tau^*}{db} > 0, \quad \frac{d\tilde{K}}{db} > 0. \quad (21)$$

**Proof:** See Appendix F. \qed

The first part of Proposition 2 repeats the results of Hong and Smart (2010). A higher degree of profit shifting to the tax haven raises the optimal statutory tax rate. The effect on the capital stock depends on whether the higher $b$ lowers the user cost by more than the increase in the statutory tax rate raises it. When the economy is static, the latter effect dominates for $\tilde{\tau} < 1/2$.

However, as long as the convergence to steady state is not immediate, the second result may reverse its sign. When the redistribution motive of the government is sufficiently weak, i.e., when $\beta \in [\hat{\beta}, 1]$, it increases the statutory tax rate by less, such that the overall impact of more internal borrowing on the user cost of capital is negative. Hence, the capital stock is higher in the new equilibrium. Unfortunately, no unambiguous results can be derived for lower values of $\beta$. However, Proposition 1 highlights the qualitative role that the convergence rate plays for the relationship between the amount of internal debt and the capital stock. Only a small deviation from the case of immediate convergence makes the relationship ambiguous.

I turn next to the impact of profit shifting on welfare. The focus is on two effects: the short term impact (i.e., the change in $\Omega_0$) and the long term impact (i.e., the change in $\tilde{\Omega}$). While the relationship between internal debt and welfare is, in general, indeterminate, there are two special cases that lead to unambiguous results. The next Proposition summarizes these results:

**Proposition 2.** Suppose the economy is initially in a steady state and internal debt $b$ increases by $db > 0$ in period 0. Then, the following results hold:

(a) If $\mu_1 \to -\infty$, such that the economy is static, welfare increases both in the short and the long term, if $\beta < 1$, and remains unchanged, if $\beta = 1$:

$$\frac{d\Omega_0}{db} = \frac{d\tilde{\Omega}}{db} = (1 - \beta)\left(G(D, \tilde{L}^d) - \tilde{w}\tilde{L}^d\right) \frac{\tau^*r}{F_K - br} \geq 0. \quad (22)$$

(b) Suppose the economy is not static, i.e., $\mu_1 \in ]-\infty, 0[$. Then, there exists $\underline{\beta} \in [0, 1]$ such that for $\beta \in [\underline{\beta}, 1]$, the welfare change in nonmonotonous with a negative short term and a positive long term change:

$$\frac{d\Omega_0}{db} < 0, \quad \frac{d\tilde{\Omega}}{db} > 0. \quad (23) \quad (24)$$
Proof: See Appendix G.

Part (a) of Proposition 2 restates the result from the static model (see Hong and Smart, 2010), and coincides with their Proposition 4. However, Appendix G shows that this result is, in general, not unambiguous. When the economy cannot immediately reach the steady state, the capital stock adjusts only gradually to the change in $b$. Moreover, the speed of convergence of the economy affects the response of the tax rate to a change in internal debt, which additionally impacts the transition of the capital stock.

Part (b) of Proposition 2 states that, for a sufficiently weak redistribution motive, there is an unambiguous negative short term effect of more internal debt on welfare, while the long-term impact is positive. Hence, welfare responds nonmonotonically to an increase in profit shifting. The intuition behind this results is the following. In the short term, the capital stock is fixed and the only welfare effects are the direct negative impact of $b$ on the tax revenues and the initial change in the statutory tax rate. Even though the government finds it optimal to increase its tax rate immediately, this policy cannot compensate for the direct loss of tax revenues in period 0 for a sufficiently high $\beta$ and welfare declines. During the transition period, the capital stock increases (see Proposition 1) which raises welfare. In the long-term, the positive impact of more investment overcompensates the initial negative welfare change. Hence, in the long-term welfare improves.

Even though both cases (a) and (b) predict a positive long term welfare change, the intuition behind these results is different. On the one hand, in case (a), the capital stock declines in the long-run. Nevertheless, welfare increases due to the sufficiently strong increase in the statutory tax rate. On the other hand, in case (b), the capital stock increases in the long-run and compensates for the loss of tax revenues and the insufficient increase in the statutory tax rate.

Proposition 2 hints at an ambiguous overall welfare impact of a change in profit shifting. However, the next Proposition shows that the overall welfare impact is actually determinate:

**Proposition 3.** Suppose the economy is initially in steady state and internal debt $b$ increases by $db > 0$ in period 0. The overall impact on welfare is unambiguously nonnegative:

$$\frac{d}{db} \int_0^\infty \Omega_t e^{-rt} dt = \frac{\tau^*(1-\beta)(G(D, \tilde{L}^d) - \tilde{w}\tilde{L}^d)}{F_K - rb} \geq 0.$$  \hspace{1cm} (25)

Proof: See Appendix H.

Hence, internal debt has an unambiguously nonnegative impact on overall welfare. The effect is strictly positive for $\beta < 1$. While Proposition 3 coincides with the result of
Hong and Smart (2010), it once again highlights the importance of analyzing internal debt is a dynamic setting. It shows that while the static model is correct in its overall welfare analysis, it misses the possible nonmonotonic welfare reaction (Proposition 2), as well as the possible positive relation between internal debt and the capital stock (Proposition 1). Therefore, it is not sufficient to derive the overall welfare impact of profit shifting to understand how the economy reaches the new steady state.

5 Time-varying tax rate

In the previous section, I derived the welfare implications of profit shifting under the assumption of a constant tax rate. In this section, I analyze whether this constraint on the government’s policy has a qualitative impact on the main results, summarized in Propositions 1, 2 and 3.

Assume the tax rate is time-dependent. Thus, the period $t$ statutory tax is denoted by $\tau_t$. The profit-maximizing first-order conditions of the national and multinational firms are again given by Equations (2), (7) and (8) with the only difference that now the tax rate $\tau$ is replaced by $\tau_t$ in Equation (7).

To solve for the government’s optimal tax policy in this section, one needs to derive the impact of a change in the tax rate $\tau_t$ on investment $I_t$. To do so, it is convenient to first define $L^m_t, L^d_t, w_t$ and $I_t$ as implicit functions of the capital stock $K_t$. First, $L^m_t, L^d_t, w_t$ are again defined implicitly as $L^m_t \equiv \ell^m_t(K_t), L^d_t \equiv \ell^d_t(K_t), w_t \equiv \omega_t(K_t)$ by Equations (2), (8) and (9) with derivatives with respect to $K_t$ given by Equation (11). To derive $I_t$ as an implicit function of the capital stock $K_t$, I follow Turnovsky (1997), Chapter 3 and linearize $\dot{K}_t, \dot{I}_t$ near the steady state. In Appendix I, I derive the following result:

**Lemma 3.** Suppose the firm is near its steady state in period $t$ and expects the tax rate $\tau_t$ to remain constant. Then, the values of capital and investment in periods $s \geq t$ for a given initial capital stock $K_t$ are

$$K_s = \tilde{K} + (K_t - \tilde{K})e^{\hat{\mu}_1(s-t)}, \tag{26}$$

$$I_s = \frac{\hat{\mu}_1}{\tilde{K}}(K_t - \tilde{K})e^{\hat{\mu}_1(s-t)}, \tag{27}$$

where $\hat{\mu}_1$ is the speed of convergence to the steady state and is given by

$$\hat{\mu}_1 = \frac{r - \sqrt{r^2 - 4(1-\tau_t)F_{KK}G_{LL}\tilde{K}}}{2C''(0)(F_{LL} + G_{LL})} < 0. \tag{28}$$

**Proof:** See Appendix I.
On the one hand, the MNE is forward-looking and the solution for investment in \((27)\) depends on the steady state that the firm expects to achieve. On the other hand, the firm is partly myopic as it also expects the tax rate to remain unchanged in the periods \(s \geq t\). This assumption is, however, less restrictive than it seems because Equations \((26)\) and \((27)\) are derived when the economy is near its steady state, i.e., when \(\tau_t\) is approximately equal to its steady state value \(\bar{\tau}\). Moreover, the speed of convergence \(\hat{\mu}_1\) differs from \(\mu_1\) (as defined in Lemma 1) only by its dependence on the tax rate \(\tau_t\) instead of the constant tax rate \(\tau\).

Define investment as a function of the capital stock and the statutory tax rate in period \(t\) as \(I_t \equiv \iota_t(K_t, \tau_t)\). Using Lemma 3, I derive in Appendix I the following results:

\[
\frac{dI_t}{dK_t} = \frac{\hat{\mu}_1}{K} < 0, \quad (29)
\]

\[
\frac{dI_t}{d\tau_t} = -\frac{\hat{\mu}_1(F_K(K_t, l_t^n) - rb)(F_{LL} + G_{LL})}{(1 - \tau_t)F_{KK}G_{LL}K} < 0. \quad (30)
\]

According to \((29)\), a higher capital stock in period \(t\), holding the steady state capital stock \(\bar{K}\) unchanged, is associated with lower investment in the same period, as less investment is required to reach the steady state. Moreover, a higher tax in period \(t\) lowers investment immediately, as shown by Equation \((30)\).

The government maximizes the same objective function as in Section 3. Then, it solves

\[
\max_{\tau_t} \int_0^\infty \Omega_t e^{-\rho t} dt \quad \text{s. t.} \quad \dot{K}_t = I_t K_t \quad (31)
\]

and taking into account \(I_t = \iota_t(K_t, \tau_t), L_t^n = l_t^n(K_t), L_t^d = l_t^d(K_t)\) and \(w_t = \omega_t(K_t)\). The optimal policy is summarized in the following proposition:

**Proposition 4.** The optimal tax rate \(\tau_t\) is an increasing function of the capital stock \(K_t\), given by

\[
\tau_t = \bar{\tau} + \frac{a_{23}}{\hat{\mu}_1} \left( K_0 - \bar{K} \right) e^{\hat{\mu}_1 t}, \quad (32)
\]

where \(a_{23} < 0\) is defined in Equation \((J.20f)\) in Appendix J, while \(\bar{\tau}\) is the optimal tax rate in steady state and is equal to \(\tau^*\) from Equation \((18)\).

**Proof:** See Appendix J.

According to Proposition 4, the optimal tax in steady state is the same as the optimal tax in the model with a constant tax rate. The intuition is that once the economy is in steady state, the government finds it optimal to levy a constant tax rate which then coincides with \(\tau^*\). However, during transition the the optimal tax is increasing in the capital stock \(K_t\).
Therefore, the government’s response to an increase in profit shifting in the steady state is identical to its reaction in the model with a static tax rate. Hence, Proposition 1 can be derived analogously by replacing $\tau^*$ with $\tilde{\tau}$. Moreover, the fact that policy responds identically in the long term means that the steady state welfare impact of a change in internal debt is the same as under a static tax rate (i.e., Equations (22) and (23) from Proposition 2 continue to hold). The short-term welfare effect may, however, differ. If $d\bar{K}/db > 0$, then the tax rate in the period of the shock $\tau_0$ increases by less than $\tilde{\tau}$ (see Equation (32)). The reverse is true in the case $d\bar{K}/db < 0$. In the case of weak redistribution preferences ($\beta$ sufficiently close to one), the long-term capital stock is increasing in internal debt (according to Proposition 1) and, thus, the initial change in the tax rate is less than the long-term change: $d\tau_0 < d\tilde{\tau}$. This effect worsens the negative initial welfare impact in the case of a large $\beta$. I summarize these results in the next proposition, while the formal proof is relegated to Appendix K:

**Proposition 5.** In the model with a time-varying tax rate, Proposition 1 holds when one replaces $\tau^*$ with $\tilde{\tau}$. Propositions 2 and 3 remain qualitatively unchanged.

**Proof:** See Appendix K. 

Hence, the main results are robust to an extension with a time-varying tax rate. One limitation of the extension is the assumption of partially myopic firms (see Lemma 3). An analysis where firms fully anticipate future tax rate changes is left to future research.

6 Conclusions

Despite the high policy relevance and large body of research on the effects of tax havens on welfare in non-haven countries, there is no consensus in the economic literature on the sign of these effects. A positive impact is usually found when one considers that tax haven operations increase the return to investment in high-tax jurisdictions (Hong and Smart, 2010; Desai et al., 2006b).

I develop a dynamic model to take explicitly into account that capital is less mobile in the short-run compared to the long-run. The consideration of transitional dynamics may affect qualitatively both the relation between internal debt and capital accumulation and the short term welfare impact of profit shifting. If the redistribution motive of the government in the high-tax country is not too strong, the short term welfare change is unambiguously negative. While overall welfare is increasing in the degree of international tax planning, the intuition behind this positive result depends may be different in the static and dynamic models.

Hence, the results of this paper are of importance to theoretical work that tries to explain the relation between profit shifting, capital accumulation and welfare. It is
also important to the empirical evaluation of profit shifting. A reform that affects the
degree of international tax planning may be evaluated as having as positive or negative
impact depending on how much time has passed between the reform and the time of
analysis. Lastly, the paper highlights the need to identify and address the possible
short-run negative welfare effects of profit shifting.

This paper’s results open multiple directions for future research. While I use the
framework of Hong and Smart (2010) to make my results comparable to theirs, future
research should conduct the dynamic analysis in other frameworks, e.g., in a representa-
tive agent model. Moreover, MNEs use other channels to shift profits to tax havens,
such as transfer price manipulation. Incorporating these channels is a possible line of
extension that may affect the results as in the static model of Gresik et al. (2015).
Furthermore, I do not explicitly model the savings behavior of agents. Including it and
endogenizing the interest rate in the model may have important implications.

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A Derivation of the MNE’s maximization problem

The MNE issues new equity $E_t$ in period $t$ with a price $q_t$ on the world capital market. The value of this equity is, thus, $q_tE_t$. Investors purchasing this equity earn next period dividends $D_t$ and capital gains $\dot{q}_tE_t$. They can, however, also invest in other assets in the world market and earn the interest rate $r$. They are indifferent between investing in the MNE and earning $r$ if

$$\frac{D_t + \dot{q}_tE_t}{q_tE_t} = r. \quad (A.1)$$

Differentiate the value of equity $V_t = q_tE_t$ with respect to time:

$$\dot{V}_t = \dot{q}_tE_t + q_t\dot{E}_t. \quad (A.2)$$

Our objective is to solve (A.2) for the value of equity $V_t$. Note that the net profit $\tilde{\pi}_t^M$ can either be used to pay dividends $D_t$ or be held as retained earnings $RE_t$, such that

$$\tilde{\pi}_t^M = D_t + RE_t. \quad (A.3)$$

Furthermore, new investment can be financed either through retained earnings $RE_t$ or new equity issues $q_t\dot{E}_t$. Thus, we have

$$(I_t + C(I_t))K_t = RE_t + q_t\dot{E}_t. \quad (A.4)$$

Use Equations (A.3),(A.4), (3) and (5) to solve for $q_t\dot{E}_t$:

$$q_t\dot{E}_t = (I_t + C(I_t))K_t - RE_t = -\pi_t^M + D_t. \quad (A.5)$$

Inserting (A.5) in (A.2), we get

$$\dot{V}_t = \dot{q}_tE_t - \pi_t^M + D_t. \quad (A.6)$$

Next, solve (A.1) for $\dot{q}_tE_t$ and insert the resulting expression in (A.6) to get

$$\dot{V}_t = rq_tE_t - \pi_t^M = rV_t - \pi_t^M. \quad (A.7)$$

The solution of the differential equation (A.7) is Equation (6).
B  The MNE optimization problem

The MNE maximizes Equation (6) over \( I_t \) and \( L^m_t \) subject to the equation of motion (4) and an initial condition \( K(0) = K_0 \). Its Hamiltonian is given by

\[
H = \pi^M_t + \lambda_t I_t K_t, \tag{B.1}
\]

where \( \lambda_t \) is the co-state variable. The first-order conditions are

\[
\frac{\partial H}{\partial L^m_t} = (1 - \tau)(F_L(K_t, L_t) - w_t) = 0, \tag{B.2}
\]

\[
\frac{\partial H}{\partial I_t} = -(1 - C'(I_t))K_t + \lambda_t K_t = 0, \tag{B.3}
\]

\[
\lambda_t r - \dot{\lambda}_t = \frac{\partial H}{\partial K} = (1 - \tau)F_K(K_t, L_t) - I_t - C(I_t) + \tau rb + \lambda_t I_t. \tag{B.4}
\]

Equation (B.3) gives \( \lambda_t = 1 + C'(I_t) \). Differentiation of this equation with respect to time results in the expression \( \dot{\lambda}_t = C''(I_t) \). Inserting the expressions for \( \lambda_t \) and \( \dot{\lambda}_t \) in (B.4) leads to Equation (7). Equation (8) follows directly from (B.2).

C  Proof of Lemma 1

To perform the comparative dynamic analysis, I follow Wildasin (2003). Suppose that at time \( t \), the government increases the tax rate by \( d\tau > 0 \).

First, derive the impact on the steady state capital stock \( \tilde{K} \), given by \( d\tilde{K}/d\tau \) (Equation (14)). Differentiate totally Equation (10a) with respect to \( \tilde{K} \) and \( \tau \), taking into account that \( \tilde{L} = \ell^{m}(\tilde{K}) \) according to Equation (11). The resulting expression is

\[
\left[ F_{KK} + F_{KL} \frac{d\ell^{m}}{dK} \right] d\tilde{K} = \frac{-r(1 - \tau) - r(1 - b\tau)(-1)}{(1 - \tau)^2} d\tau,
\]

\[
\Rightarrow \frac{d\tilde{K}}{d\tau} = \frac{r(1 - b)(F_{LL} + G_{LL})}{(1 - \tau)^2 F_{KK} G_{LL}} \tag{C.1}
\]

Equation (C.1) coincides with (14) in Lemma 1.

To derive \( dK_t/d\tau \), differentiate the first-order conditions (B.3) and (B.4) together with the equation of motion (4) with respect to \( \tau, K_t, \dot{K}_t, \lambda_t, \dot{\lambda}_t \) and \( I_t \):

\[
d\lambda_t = C''(I_t) dI_t \tag{C.2}
\]

\[
d\lambda_t(r - I) - d\dot{\lambda}_t = (1 - \tau) \left[ F_{KK} + F_{KL} \frac{d\ell^{m}}{dK} \right] dK_t + (\lambda_t - 1 - C'(I_t)) dI_t - \left[ F_K(K_t, L^m_t) - r b \right] d\tau \tag{C.3}
\]

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\[ d\dot{K}_t = I_t dK_t + K_t dI_t. \] (C.4)

Suppose that the economy is near steady state with \( K_t \approx \tilde{K}, I_t \approx \tilde{I} = 0 \). Then, Equations (C.4) and (C.2) become

\[ d\dot{K}_t = \tilde{K} dI_t, \] (C.5)
\[ d\dot{\lambda}_t = C''(0) dI_t = C''(0) \frac{d\dot{K}_t}{\tilde{K}}. \] (C.6)

Moreover, we can differentiate Equation (C.6) with respect to time, which gives

\[ d\ddot{\lambda}_t = C''(0) d\ddot{K}_t. \] (C.7)

One can now use Equations (B.3), (C.6), (C.7) and (11) to simplify (C.3):

\[ \frac{d\dot{K}_t}{d\tau} - r \frac{d\dot{K}_t}{d\tau} + \frac{(1 - \tau) F_{KK} G_{LL} \tilde{K}}{C''(0)(F_{LL} + G_{LL})} dK_t d\tau = \frac{\tilde{K}(F_K - rb)}{C''(0)} \]. (C.8)

Equation (C.8) is a second-order heterogeneous differential equation in \( d\dot{K}_t/d\tau \).

The particular solution to (C.8) is found by setting \( d\ddot{K}_t = d\dot{K}_t = 0 \). Thus, the particular solution is

\[ \frac{dK_t}{d\tau} = \frac{(F_K - rb)(F_{LL} + G_{LL})}{(1 - \tau) F_{KK} G_{LL}} \]. (C.9)

To find solution to the homogeneous part (i.e., the left-hand side) of (C.8), we suppose that the solution is of the form \( d\dot{K}_t/d\tau = Ae^{\mu t} \), where \( A \) is an undetermined constant. Under the exponential functional form, we have \( d\dot{K}_t/d\tau = \mu dK_t/d\tau \) and \( d\ddot{K}_t/d\tau = \mu^2 dK_t/d\tau \). Hence, the homogeneous part of (C.8) can be rewritten as

\[ \mu^2 - r\mu + \frac{(1 - \tau) F_{KK} G_{LL} \tilde{K}}{C''(0)(F_{LL} + G_{LL})} = 0. \] (C.10)

Equation (C.10) has two solutions for \( \mu \), given by

\[ \mu_1 = \frac{r - \sqrt{r^2 - \frac{4(1 - \tau) F_{KK} G_{LL} \tilde{K}}{C''(F_{LL} + G_{LL})}}}{2} < 0, \quad \mu_2 = \frac{r + \sqrt{r^2 - \frac{4(1 - \tau) F_{KK} G_{LL} \tilde{K}}{C''(F_{LL} + G_{LL})}}}{2} > 0. \] (C.11)

Therefore, Equation (C.10) one positive and one negative root. The solution to the homogeneous part is, thus,

\[ \frac{dK_t}{d\tau} = A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t}, \] (C.12)
where \( A_1 \) and \( A_2 \) are undetermined coefficients. The general solution is the sum of the homogeneous and particular solutions:

\[
\frac{dK_t}{d\tau} = \frac{(F_K - rb)(F_{LL} + G_{LL})}{(1 - \tau)F_{KK}G_{LL}} + A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t}.
\]

(C.13)

Invoking the initial condition \( \frac{dK_0}{d\tau} = 0 \) and the terminal condition \( \lim_{t \to \infty} \frac{dK_t}{d\tau} = \frac{d\tilde{K}}{d\tau} \), one gets \( A_1 = -\frac{d\tilde{K}}{d\tau} \) and \( A_2 = 0 \). This completes the proof of Lemma 1. \( \square \)

\[ \textbf{D} \quad \text{Derivation of the optimal tax rate (Equation (18))} \]

The government’s objective function is

\[
\int_0^\infty \Omega_t e^{-rt} dt = \int_0^\infty (C_t^W + \beta C_t^E)e^{-rt} dt
\]

\[\text{(D.1)}\]

\[
= \int_0^\infty (\tau[F(K_t, L_{m}^t) - rbK_t] + (1 - \tau)w_t L_{m}^t + G(D, L_{d}^t) - (1 - \tau)(1 - \beta)[G(D, L_{d}^t) - w_t L_{d}^t])e^{-rt} dt.
\]

It maximizes (D.1) subject to \( L_{m}^t = \ell_{m}^t(K_t), L_{d}^t = \ell_{d}^t(K_t), w_t = \omega_t(K_t) \) and Equation (12). The first-order condition is

\[
\frac{\partial}{\partial \tau} = \int_0^\infty \left\{ F(K_t, L_{m}^t) - rbK_t - w_t L_{m}^t + (1 - \beta)[G(D, L_{d}^t) - w_t L_{d}^t] + \left[ \tau(F_K - rb) + (1 - \tau)w_t \right] \frac{d\ell_{m}^t}{dK_t} + (1 - \tau)[L_{m}^t + (1 - \beta)L_{d}^t] \frac{d\omega_t}{dK_t} + [G_L - (1 - \tau)(1 - \beta)[G(D, L_{d}^t) - w_t]] \frac{d\ell_{d}^t}{dK_t} \right\} e^{-rt} dt = 0.
\]

(D.2)

Using the labor demand equations \( F_L = w \) and \( G_L = w \), we can simplify (D.2):

\[
\frac{\partial}{\partial \tau} = \int_0^\infty \left\{ F(K_t, L_{m}^t) - rbK_t - w_t L_{m}^t + (1 - \beta)[G(D, L_{d}^t) - w_t L_{d}^t] + \left[ \tau(F_K - rb) + (1 - \tau)L_{m}^t + (1 - \beta)L_{d}^t \right] \frac{d\omega_t}{dK_t} \right\} e^{-rt} dt = 0.
\]

(D.3)

Following Wildasin (2003), I assume that the economy is near its steady state, such that \( K_t \approx \bar{K}, L_{m}^t \approx \bar{L}^m, L_{d}^t \approx \bar{L}^d, w_t \approx \bar{w} \) and \( dK_t/d\tau = d\bar{K}/d\tau(1 - e^{\mu_1 t}) \). Thus, (D.3)
becomes
\[
\frac{\partial}{\partial \tau} = \int_0^\infty \left\{ F(\tilde{K}, \tilde{L}) - rb\tilde{K} - \tilde{w}\tilde{L} + (1 - \beta)[G(D, \tilde{L}) - \tilde{w}\tilde{L}] 
\right.
\]
\[
+ \left[ \tau(F_K - rb) + (1 - \tau)[\tilde{L} + (1 - \beta)\tilde{L}\frac{d\tilde{\omega}}{d\tau}] \frac{d\tilde{K}}{d\tau}(1 - e^{\mu_1 t}) \right\} e^{-rt} dt = 0,
\]

where \( d\tilde{\omega}/d\tilde{K} \) is the value of \( d\omega_t/dK_t \), when evaluated at the steady state. Use Equations (11) and (C.1) to define \( d\tilde{\omega}/d\tau \) as
\[
\frac{d\tilde{\omega}}{d\tau} = \frac{d\tilde{\omega}}{d\tilde{K}} \frac{d\tilde{K}}{d\tau} = \frac{(F_K - rb)F_{LK}}{(1 - \tau)F_{KK}}.
\]

Integrate the left-hand side of (D.4) to get
\[
0 = F(\tilde{K}, \tilde{L}) - rb\tilde{K} - \tilde{w}\tilde{L} + (1 - \beta)[G(D, \tilde{L}) - \tilde{w}\tilde{L}]
\]
\[
- \frac{\mu_1}{r - \mu_1} \left[ \tau(F_K - rb) \frac{d\tilde{K}}{d\tau} + (1 - \tau)[\tilde{L} + (1 - \beta)\tilde{L}\frac{d\tilde{\omega}}{d\tau}] \right].
\]

I use now the constant returns property of the production function \( F(\cdot) \), which allows \( F(\cdot) \) to be represented as \( F(K, L) = F_K K + F_L L = F_K K + \tilde{w}L \). Thus, (D.6) becomes
\[
0 = (F_K - rb)\tilde{K} + (1 - \beta)[G(D, \tilde{L}) - \tilde{w}\tilde{L}]
\]
\[
- \frac{\mu_1}{r - \mu_1} \left[ \tau(F_K - rb) \frac{d\tilde{K}}{d\tau} + (1 - \tau)[\tilde{L} + (1 - \beta)\tilde{L}\frac{d\tilde{\omega}}{d\tau}] \right].
\]

Moreover, the partial derivative \( F_K \) is homogeneous of degree zero, which means that
\[
0 \cdot F_K = F_{KK} K + F_{KL} L.
\]

Therefore, we get
\[
(1 - \tau)\tilde{L}\frac{d\tilde{\omega}}{d\tau} = \frac{(1 - \tau)\tilde{L}^m(F_K - rb)F_{LK}}{(1 - \tau)F_{KK}} = -(F_K - rb)\tilde{K}.
\]

Inserting (D.8) in (D.7), denoting the optimal tax as \( \tau^* \), and rearranging gives Equation (18).

\section{Proof of Lemma 2}

To prove Lemma 2, suppose that \( C''(0) = 0 \). Then, according to Equation (C.11), \( \mu_1 \to -\infty \). Hence, the economy converges immediately to steady state from any initial
capital stock $K_0$. Taking the limit of all terms in Equation (18) for $\mu_1 \to -\infty$, we get
\[
(1 - \beta) \left[ G(D, \bar{L}^d) - \bar{w}\bar{L}^d + (1 - \tau)\bar{L}^d d\bar{\omega} d\tau \right] + \tau (F_K(\bar{K}, \bar{L}^m) - rb) \frac{d\bar{K}}{d\tau} = 0. \tag{E.1}
\]
To express (E.1) in a form comparable to the optimal tax rate from Hong and Smart (2010), use their notation and define $\pi \equiv G(D, \bar{L}^d) - \bar{w}\bar{L}^d$ and $\rho \equiv r(1 - b\tau)/(1 - \tau)$, such that in steady state $F_K = \rho$ and. Now, make the following rearrangement:
\[
\tau (F_K(\bar{K}, \bar{L}^m) - rb) = \bar{\tau}(\rho - rb) = \frac{\bar{\tau}r(1 - b)}{(1 - \tau)} = \rho - r. \tag{E.2}
\]
Equations (E.1), (E.2) and the definition of $\pi$ together give
\[
(1 - \beta) \left[ \pi + (1 - \tau)\bar{L}^d d\bar{\omega} + (1 - \tau)(\rho - r) \frac{d\bar{K}}{d\tau} \right] = 0. \tag{E.3}
\]
Hong and Smart (2010) take $\rho$ as the strategic variable of the government. Therefore, multiply (E.3) by $d\tau/d\rho$ to get
\[
(1 - \beta) \left[ \frac{\pi d\tau}{d\rho} + (1 - \tau)\bar{L}^d d\bar{\omega} + (1 - \tau)(\rho - r) \frac{d\bar{K}}{d\rho} \right] = 0. \tag{E.4}
\]
Equation (E.4) is identical to Equation (16) from Hong and Smart (2010) that determines the equilibrium tax rate in the static model.

\[\text{F Proof of Proposition 1}\]

The effects of a change in $b$ on $\tau^*$ and $\bar{K}$ can be derived from Equations (10a) and (D.6), which determine the steady state capital stock and optimal tax rate, respectively. We can express these equations as
\[
0 = F_K(\bar{K}, \bar{L}^m)(1 - \tau^*) - (1 - b\tau^*), \tag{F.1}
\]
\[
0 = F(\bar{K}, \bar{L}^m) - br\bar{K} - \bar{w}\bar{L}^m + (1 - \beta)[G(D, \bar{L}^d) - \bar{w}\bar{L}^d] - \frac{\mu_1}{r - \mu_1} \left[ \tau^*(F_K(\bar{K}, \bar{L}^m) - rb) + (1 - \tau^*)(\bar{L}^m + (1 - \beta)\bar{L}^d) \frac{d\bar{\omega}}{d\bar{K}} \right] \frac{d\bar{K}}{d\tau}. \tag{F.2}
\]
The next step is to differentiate totally (F.1) and (F.2) with respect to $\bar{K}, \tau^*$ and $b$. Note, first, that $\mu_1$ depends on $\tau$ and $\bar{K}$ in the following way (in the following
derivations, we neglect the third-order derivatives of the production function):

\[
\frac{d\mu_1}{d\tau} = \frac{-F_{KKK}G_{LL}\tilde{K}}{C''(0)(F_{LL} + G_{LL})\sqrt{\tau^2 - \frac{4(1-\tau)F_{KKK}G_{LL}\tilde{K}}{C''(F_{LL} + G_{LL})}}} = -\frac{\mu_1(r - \mu_1)}{(r - 2\mu_1)(1 - \tau)}, \quad (F.3)
\]

\[
\frac{d\mu_1}{dK} = \frac{(1 - \tau)F_{KKK}G_{LL}}{C''(0)(F_{LL} + G_{LL})\sqrt{\tau^2 - \frac{4(1-\tau)F_{KKK}G_{LL}\tilde{K}}{C''(F_{LL} + G_{LL})}}} = \frac{\mu_1(r - \mu_1)}{(r - 2\mu_1)\tilde{K}}, \quad (F.4)
\]

Using Equations (F.3) (F.4), as well as (11) and (14), the total differential of (F.1) and (F.2) is

\[
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
  d\tilde{K} \\
  d\tau^*
\end{pmatrix}
= \begin{pmatrix}
  -b_1 \\
  -b_2
\end{pmatrix} \, dB,
\quad (F.5)
\]

where

\[
a_{11} \equiv (1 - \tau^*)\frac{F_{KKK}G_{LL}}{F_{LL} + G_{LL}}, \quad (F.6a)
\]

\[
a_{12} \equiv -(F_K - rb), \quad (F.6b)
\]

\[
a_{21} \equiv \frac{F_K - rb}{r - \mu_1} \left[ \frac{r(1 - \tau^*) - \mu_1(1 + \tau^*)}{1 - \tau^*} + \frac{\mu_1\beta F_{LL}}{F_{LL} + G_{LL}} - \frac{r\mu_1\tau^*}{(r - 2\mu_1)\tilde{K}} \frac{d\tilde{K}}{d\tau} \right] - \frac{r}{r - \mu_1} \frac{d\tilde{\omega}}{d\tilde{K}} \left( \tilde{L}^m + (1 - \beta)\tilde{L}^d \right) \left[ 1 + \frac{\mu_1(1 - \tau^*)}{(r - 2\mu_1)\tilde{K}} \frac{d\tilde{K}}{d\tau} \right], \quad (F.6c)
\]

\[
a_{22} \equiv -\frac{\mu_1}{d\tau^*} \left[ (F_K - rb)[r(1 - \tau^*) - 2\mu_1] - r(1 - \tau^*) \frac{d\tilde{\omega}}{d\tilde{K}} \left( \tilde{L}^m + (1 - \beta)\tilde{L}^d \right) \right], \quad (F.6d)
\]

\[
b_1 \equiv r\tau^*, \quad (F.6e)
\]

\[
b_2 \equiv -r \left[ \tilde{K} - \frac{\mu_1(F_{LL} + G_{LL})}{(r - \mu_1)(1 - \tau^*)F_{KKK}G_{LL}} \left( 2\tau^*(F_K - rb) + (1 - \tau^*) \frac{d\tilde{\omega}}{d\tilde{K}} \left( \tilde{L}^m + (1 - \beta)\tilde{L}^d \right) \right) \right]. \quad (F.6f)
\]

The determinant of the matrix \( J \) is given by

\[
|J| = a_{11}a_{22} - a_{12}a_{21} = \left\{ \frac{(F_K - rb)}{1 - \tau^*} \left[ (r - 2\mu_1) \left( r(1 - \tau^*) - \mu_1(2 + \tau^*) + \mu_1(1 - \tau^*)\beta \frac{F_{LL}}{F_{LL} + G_{LL}} \right) + \mu_1\tau^* \left( 1 - \frac{1 - \tau^* d\tilde{K}}{K} \right) \right] \right\} \quad (F.7)
\]
Using Cramer’s rule, the effects of $b$ on the steady state capital stock and tax rate are:

$$
\frac{d\tilde{K}}{db} = \frac{1}{|J|} \begin{vmatrix} -b_1 & a_{12} \\ -b_2 & a_{22} \end{vmatrix} = \frac{b_2a_{12} - b_1a_{22}}{|J|}
$$

$$
= \frac{r(F_K - rb)}{|J|(r - 2\mu_1)(r - \mu_1)(1 - \tau^*)} \left\{ \mu_1\tau^* \frac{d\tilde{K}}{d\tau} \left[ 2\mu_1(1 - 2\tau^*) - r(1 - \tau^*) \right] + \tilde{K} \left[ r(1 - \tau^*) - \mu_1(2 - 3\tau^*) + \mu_1(1 - \beta) \frac{\tilde{L}^d}{L^m}(r - 2\mu_1(1 - \tau^*)) \right] \right\}
$$

$$
\frac{d\tau^*}{db} = \frac{1}{|J|} \begin{vmatrix} a_{11} & -b_1 \\ a_{21} & -b_2 \end{vmatrix} = \frac{b_1a_{21} - b_2a_{11}}{|J|}
$$

$$
= \frac{r}{|J|(r - \mu_1)} \left\{ (F_K - rb)\tau^* \left[ r - \mu_1 \left( 1 - \beta \frac{F_{LL}}{F_{LL} + G_{LL}} \right) - \frac{2\mu_1}{1 - \tau^*} \right] - \frac{r\mu_1\tau^*}{(r - 2\mu_1)\tilde{K}} \frac{d\tilde{K}}{d\tau} \right\} + \frac{F_{KK}G_{LL}\tilde{K}}{F_{LL} + G_{LL}} \left[ r + (1 - \beta) \frac{\tilde{L}^d}{L^m}(1 - \tau^*) \right]
$$

$$
+ \left( 1 + (1 - \beta) \frac{\tilde{L}^d}{L^m} \right) \frac{r\mu_1\tau^*(1 - \tau^*)}{(r - 2\mu_1)\tilde{K}} \frac{d\tilde{K}}{d\tau} \right\}.
$$

The expression (F.8) contains only negative terms in its first row and both negative and positive terms in the second row. The derivative of $\tilde{K}$ with respect to $b$ has, thus, an ambiguous sign. The same holds true for (F.9): while the first row of (F.9) is positive, the second and third rows are either positive or negative.

To prove the first part of Proposition 1, take the limits of (F.8) and (F.9) when $\beta$ approaches $-\infty$:

$$
\lim_{\mu_1 \to -\infty} \frac{d\tilde{K}}{db} = \frac{r(F_K - rb)}{|J|(1 - \tau^*)} \left[ \tau^* \frac{d\tilde{K}}{d\tau} (1 - 2\tau^*) - \tilde{K}(1 - \beta) \frac{\tilde{L}^d}{L^m}(1 - \tau^*) \right] < 0, \quad \text{if} \quad \tau^* < \frac{1}{2},
$$

$$
\lim_{\mu_1 \to -\infty} \frac{d\tau^*}{db} = \frac{-r}{|J|} \left\{ -(F_K - rb)\tau^* \left[ 1 - \beta \frac{F_{LL}}{F_{LL} + G_{LL}} + \frac{2}{1 - \tau^*} \right] + \frac{F_{KK}G_{LL}\tilde{K}}{F_{LL} + G_{LL}} (1 - \beta) \frac{\tilde{L}^d}{L^m}(1 - \tau^*) \right\} > 0.
$$

To proof the second part of Proposition 1, evaluate (F.8) and (F.9) at $\beta = 1$. Note first
that, in this case, the optimal tax rate is determined by

\[
\frac{\tau^* d\tilde{K}}{d\tau} \frac{d\tilde{K}}{K} = \frac{r}{\mu_1},
\]

where (F.12) is Equation (18), evaluated at \( \beta = 1 \). Evaluating Equations (F.8) and (F.9) at \( \beta = 1 \) and using (F.12), one gets

\[
d\tilde{K}_{db}(\beta = 1) = -\frac{r^2 \mu_1 \tau^* \tilde{K}}{J|(r - 2\mu_1)(r - \mu_1)(1 - \tau^*)} > 0,
\]

\[
d\tau^*_{db}(\beta = 1) = \frac{r}{J|(r - \mu_1)} \left[ -(F_K - rb)\tau^* \mu_1 \left( \frac{2r}{r - 2\mu_1} + \frac{G_{LL}}{F_{LL} + G_{LL}} + \frac{2}{1 - \tau^*} \right) \right. \\
\left. + \frac{F_{KK}G_{LL}\tilde{K}r(r(2 - \tau^*) - 2\mu_1)}{(F_{LL} + G_{LL})(r - 2\mu_1)} \right].
\]

Equation (F.13) is unambiguously positive. To derive the sign of (F.14), rewrite (F.12) using (C.1):

\[
\frac{F_{KK}G_{LL}\tilde{K}r}{F_{LL} + G_{LL}} = \frac{\tau^*(F_K - br)\mu_1}{(1 - \tau^*)}.
\]

Inserting (F.15) in (F.16) and simplifying, one gets

\[
d\tau^*_{db}(\beta = 1) = -\frac{r(F_K - rb)\tau^* \mu_1 \left[ r(2 - \tau^*) + \frac{(r - 2\mu_1)(1 - \tau^*)G_{LL}}{F_{LL} + G_{LL}} - 2\mu_1 \right]}{|J|(r - \mu_1)(r - 2\mu_1)(1 - \tau^*)} > 0.
\]

Together, Equations (F.13) and (F.16) prove the second part of Proposition 1 for \( \beta = 1 \). Since both (F.8) and (F.9) are continuous in \( \beta \) there exist values of \( \beta \) close to but not equal to one for which \( d\tilde{K}/db > 0 \) and \( d\tau^*/db > 0 \). Denote the lowest value of \( \beta \) for which these relations hold as \( \hat{\beta} \). Then, for \( \beta \in [\hat{\beta}, 1] \), the steady state capital stock and tax rate are increasing in the amount of internal debt.

Lastly, note that for all values of \( \beta \) not yet considered, i.e., \( \beta \in [0, \hat{\beta}] \), the effects of an increase in \( b \) are given by (F.8) and (F.9) and are ambiguous.

\[\square\]

**G Proof of Proposition 2**

To prove Proposition 2, begin by expressing the steady state welfare as

\[\tilde{\Omega} = \tau^*[F(\tilde{K}, \tilde{L}^m) - rb\tilde{K}] + (1 - \tau^*)\tilde{w}\tilde{L}^m + G(D, \tilde{L}^d) - (1 - \tau^*)(1 - \beta)[G(D, \tilde{L}^d) - \tilde{w}\tilde{L}^d].\]
I derive first the effects of a change in $b$ on $\tilde{\Omega}$. Differentiate the welfare with respect to $b$, taking into account the effects of $b$ on $\tau^*$ and $\tilde{K}$. The resulting expression is

\[
\frac{d\tilde{\Omega}}{db} = -r\tau^*\tilde{K} + \left[ F(\tilde{K}, \tilde{L}^m) - rb\tilde{K} - \tilde{w}\tilde{L}^m + (1 - \beta)(G(D, \tilde{L}^d) - \tilde{w}\tilde{L}^d) \right] \frac{d\tau^*}{db} \\
+ \left[ \tau^*(F_K - rb) + (1 - \tau^*)\frac{d\tilde{\omega}}{dK}(\tilde{L}^m + (1 - \beta)\tilde{L}^d) \right] \frac{d\tilde{K}}{db},
\]

(G.2)

Use the government’s first-order condition (D.6) to substitute for the term in brackets in the first row of (G.2). Equation (G.2) becomes

\[
\frac{d\tilde{\Omega}}{db} = -r\tau^*\tilde{K} + \left[ \tau^*(F_K - rb) + (1 - \tau^*)\frac{d\tilde{\omega}}{dK}(\tilde{L}^m + (1 - \beta)\tilde{L}^d) \right] \left[ \frac{\mu_1}{r - \mu_1} \frac{d\tilde{K}}{d\tau} \frac{d\tau^*}{db} + \frac{d\tilde{K}}{db} \right]
\]

(G.3)

Now, I split the effect of $b$ on $\tilde{K}$ in a direct and an indirect effect:

\[
\frac{d\tilde{K}}{db} = \frac{d\tilde{K}}{d\tau} \frac{d\tau^*}{db} + \frac{\partial \tilde{K}}{\partial b},
\]

(G.4)

where the direct effect $\partial \tilde{K}/\partial b$ is derived by totally differentiating Equation (10a) with respect to $\tilde{K}$ and $b$:

\[
\frac{\partial \tilde{K}}{\partial b} = -\frac{r\tau^*(F_{LL} + G_{LL})}{(1 - \tau^*)F_{KK}G_{LL}} > 0.
\]

(G.5)

Thus, the welfare change becomes

\[
\frac{d\tilde{\Omega}}{db} = -r\tau^*\tilde{K} + \left[ \tau^*(F_K - rb) + (1 - \tau^*)\frac{d\tilde{\omega}}{dK}(\tilde{L}^m + (1 - \beta)\tilde{L}^d) \right] \left[ \frac{\mu_1}{r - \mu_1} \frac{d\tilde{K}}{d\tau} \frac{d\tau^*}{db} + \frac{d\tilde{K}}{db} \right]
\]

(G.6)
Use the government’s first-order condition (D.7) to substitute the first term in brackets in (G.6) and simplify further (G.6):

\[
\frac{d\tilde{\Omega}}{db} = -r \left[ \tau^* \tilde{K} - (r - \mu_1) \frac{(F_K - rb)\tilde{K} + (1 - \beta)(G(D, \tilde{L}^d) - \tilde{w}\tilde{L}^d)}{\mu_1 \frac{dK}{dr}} \right]
\]

\[
= -r \left[ \tau^* \tilde{K} - \frac{(F_K - rb)\tilde{K} + (1 - \beta)(G(D, \tilde{L}^d) - \tilde{w}\tilde{L}^d)}{\mu_1} \left( \frac{d\tau^*}{db} + \frac{(r - \mu_1)\partial \tilde{K}}{r \partial b} \right) \right]
\]

\[
= -r \left[ \tau^* \tilde{K} - \frac{(F_K - rb)\tilde{K} + (1 - \beta)(G(D, \tilde{L}^d) - \tilde{w}\tilde{L}^d)}{\mu_1} \left( \frac{d\tau^*}{db} - \tau^* (r - \mu_1) \frac{1}{F_K - rb} \right) \right]
\]

\[
= -r \left[ \tau^* \tilde{K} + \tau^* (r - \mu_1) \frac{1}{F_K - rb} (1 - \beta)(G(D, \tilde{L}^d) - \tilde{w}\tilde{L}^d) \right.
\]

\[
- \left. \left( (F_K - rb)\tilde{K} + (1 - \beta)(G(D, \tilde{L}^d) - \tilde{w}\tilde{L}^d) \right) \frac{d\tau^*}{db} \right]
\]

(G.7)

The first row of the last expression in (G.7) is positive, while the second row is negative for \( \frac{d\tau^*}{db} > 0 \). Hence, the net change in \( \tilde{\Omega} \) is indeterminate.

Focus first on the case \( \mu_1 \to -\infty \). Evaluate (G.7) at \( \mu_1 \to -\infty \) to get

\[
\lim_{\mu_1 \to -\infty} \frac{d\tilde{\Omega}}{db} = \frac{\tau^* r}{F_K - rb} (1 - \beta)(G(D, \tilde{L}^d) - \tilde{w}\tilde{L}^d) > 0.
\]

(G.8)

Hence, when the economy is always in steady state, the long-term impact of an increase in \( b \) on welfare is positive. Moreover, in the case \( \mu_1 \to -\infty \), the economy is static and \( d\tilde{\Omega}_0 = d\tilde{\Omega} \). This concludes the proof of part (a) of Proposition 2.

Consider now part (b) of Proposition 2. Evaluate Equation (G.7) at \( \beta = 1 \)

\[
\frac{d\tilde{\Omega}}{db} (\beta = 1) = -\frac{r\tilde{K}}{\mu_1} \left[ r\tau^* - (F_K - rb) \frac{d\tau^*}{db} \right]
\]

(G.9)

To simplify (G.9), evaluate first \(|J|\) at \( \beta = 1 \) using Equations (F.7), (F.12) and (F.15). The resulting expression is

\[
|J| \bigg|_{\beta=1} = \frac{(F_K - rb)^2 \mu_1 \left[ (4 - \tau^*) - r(3 - 2\tau^*) + \frac{(r-2\mu_1)(1-\tau^*)F_{LL}}{F_{LL}+G_{LL}} \right]}{(r - \mu_1)(r - 2\mu_1)(1 - \tau^*)} > 0.
\]

(G.10)

Inserting Equations (F.16) and (G.10) in (G.9) gives after some manipulation

\[
\frac{d\tilde{\Omega}}{db} (\beta = 1) = -\frac{[r\tau^*(F_K - rb)^2 \mu_1}{|J|(r - \mu_1)(r - 2\mu_1)(1 - \tau^*)} > 0.
\]

(G.11)

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Equation (G.11) proves Equation (24) from Proposition 2, when $\beta = 1$. Since welfare is continuous in $\beta$, this result holds also for values of $\beta$ sufficiently close but not equal to one. Define the lowest value of $\beta$ that satisfies (G.11) as $\bar{\beta}$. Then, (G.11) is satisfied for $\beta \in [\bar{\beta}, 1]$. To derive the short-term welfare change in this case, define short-term welfare $\Omega_0$ as

$$\Omega_0 = \tau_0^*[F(\tilde{K}_0, \tilde{L}_0^m) - rb\tilde{K}_0] + (1 - \tau_0^*)\tilde{w}_0\tilde{L}_0^m + G(D, \tilde{L}_0^d) - (1 - \tau_0^*)(1 - \beta)[G(D, \tilde{L}_0^d) - \tilde{w}_0\tilde{L}_0^d],$$

where a subscript 0 denotes the initial steady state. Note that the capital stock cannot change in time period zero, $d\tilde{K}_0/db = 0$, as it is a stock variable. Consequently, the wage rate and the labor demands also remain unchanged at time period 0. This initial impact on welfare of a change in internal debt in period 0 is

$$\frac{d\Omega_0}{db} = -\tau_0^*r\tilde{K}_0 + \left[F(\tilde{K}_0, \tilde{L}_0^m) - rb\tilde{K}_0 - \tilde{w}_0\tilde{L}_0^m + (1 - \beta)(G(D, \tilde{L}_0^d) - \tilde{w}_0\tilde{L}_0^d)\right]\frac{d\tau^*}{db}. \tag{G.13}$$

Evaluate (G.13) at $\beta = 1$ using the constant returns property $F = F_KK + F_LL^m$:

$$\frac{d\Omega_0}{db} (\beta = 1) = -\tau_0^*r\tilde{K}_0 + \left(F_K(\tilde{K}_0, \tilde{L}_0^m) - rb\right)\tilde{K}_0\frac{d\tau^*}{db}. \tag{G.14}$$

When the change in internal debt $db$ is small, then $\tilde{K}_0 \approx \tilde{K}$ and $\tilde{L}_0^m \approx \tilde{L}^m$, $\tau_0^* \approx \tau^*$. Hence, Equation (G.14) can be evaluated in the vicinity of $\tilde{K}, \tau^*$, which gives

$$\frac{d\Omega_0}{db} (\beta = 1) = -\tau^*r\tilde{K} + \left(F_K(\tilde{K}, \tilde{L}^m) - rb\right)\tilde{K}\frac{d\tau^*}{db}$$

$$= -\tilde{K}\left[r\tau^* - (F_K - rb)\frac{d\tau^*}{db}\right]$$

$$= \frac{\mu_1}{r}\frac{d\tilde{\Omega}}{db} < 0. \tag{G.15}$$

Equation (G.15) proves Equation (23) from Proposition 2. Following the same intuition as before, (G.15) holds for $\beta \in [\bar{\beta}, 1]$. This concludes the proof of part (b) from Proposition 2.

Lastly, note that both Equations (G.7) and (G.13) are ambiguous for $\beta < \bar{\beta}$ and $\mu_1 \in] -\infty, 0[.$
H Proof of Proposition 3

The overall welfare change is given by the effect of \( b \) on welfare, as defined in Equation (D.1). The derivative of (D.1) with respect to \( b \) is

\[
\frac{d}{db} \int_0^\infty e^{-rt} dt = \int_0^\infty \left\{ -\tau^* rK_t + \left[ F(K_t, L_t^m) - rbK_t - w_tL_t^m + (1 - \beta)[G(D, L_t^d) - w_tL_t^d] \right] \frac{d\tau^*}{db} \\
+ \left[ \tau^*(F_K - rb) + (1 - \tau^*)[L_t^m + (1 - \beta)L_t^d] \frac{d\omega_t}{dK_t} \right] \frac{dK_t}{db} \right\} e^{-rt} dt. \tag{H.1}
\]

The change in the capital stock in period \( t \) can be decomposed analogously to Equation (G.4) in a direct and indirect effect:

\[
\frac{dK_t}{db} = \frac{dK_t}{d\tau^*} \frac{d\tau^*}{db} + \frac{\partial K_t}{\partial b}. \tag{H.2}
\]

Insert (H.2) in (H.1) and rearrange to get

\[
\frac{d}{db} \int_0^\infty e^{-rt} dt = \int_0^\infty \left\{ -\tau^* rK_t + \left[ \tau^*(F_K - rb) + (1 - \tau^*)[L_t^m + (1 - \beta)L_t^d] \frac{d\omega_t}{dK_t} \right] \frac{dK_t}{db} \right\} e^{-rt} dt. \tag{H.3}
\]

The terms multiplied with \( d\tau^*/db \) sum up to the first-order condition with respect to \( \tau \) and vanish. Thus, (H.3) can be simplified to

\[
\frac{d}{db} \int_0^\infty e^{-rt} dt = \int_0^\infty \left\{ -\tau^* rK_t + \left[ \tau^*(F_K - rb) + (1 - \tau^*)[L_t^m + (1 - \beta)L_t^d] \frac{d\omega_t}{dK_t} \right] \frac{\partial K_t}{\partial b} \right\} e^{-rt} dt. \tag{H.4}
\]

Suppose that the economy is near steady state with \( K_t \approx \tilde{K} \). Then, following the same steps as in the proof of Lemma 1, one can show that

\[
\frac{\partial K_t}{\partial b} = \frac{\partial \tilde{K}}{\partial b} \left[ 1 - e^{\mu t} \right], \tag{H.5}
\]
where \( \partial \bar{K}/\partial b \) is defined in Equation (G.5). Insert Equations (11), (G.5) and (H.5) in (H.4) and evaluate the integral for \( K_t = \bar{K} \). The resulting expression is

\[
\frac{d}{db} \int_{0}^{\infty} \Omega_t e^{-rt} dt = \frac{\tau^* \bar{K}}{r - \mu_1} \left[ -r - \mu_1 (1 - \beta) \frac{\bar{L}^d}{\bar{L}^m} + \mu_1 \frac{\tau^* d\bar{K}}{d\tau} \right].
\]  

(H.6)

Now, insert Equation (D.5) in (18) and rearrange to get

\[
\left[ -r - \mu_1 (1 - \beta) \frac{\bar{L}^d}{\bar{L}^m} + \mu_1 \frac{\tau^* d\bar{K}}{d\tau} \right] = \frac{(r - \mu_1)(1 - \beta)(G(D, \tilde{d}) - \tilde{w}\tilde{L}^d)}{(F_K - rb)\bar{K}}.
\]  

(H.7)

Together Equations (H.6) and (H.7) give Equation (25) from Proposition 3.

I Proof of Lemma 3

To solve for \( K_s, I_s \) around steady state, rewrite the equation of motion for capital and the first-order condition (8) in the case of a time-varying tax rate in period \( s \geq t \) (where \( t \) is the initial period):

\[
\dot{K}_s = I_s K_s,
\]  

(I.1)

\[
\dot{I}_s = \frac{1}{C''(I_s)} \left[ r(1 - b\tau_s) + C(I_s) + C'(I_s)(r - I_s) - F_K(K_s, L^m_s)(1 - \tau_s) \right],
\]  

(I.2)

Proceed by linearizing Equations (I.1) and (I.2) around the steady state \((\bar{K}, \bar{I} = 0)\):

\[
\begin{pmatrix}
\dot{K}_s \\
\dot{I}_s
\end{pmatrix}
\approx
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
K_s - \bar{K} \\
I_s - \bar{I}
\end{pmatrix},
\]  

(I.3)

where

\[
a_{11} \equiv \frac{\partial \dot{K}_s}{\partial K_s} = I_s,
\]  

(I.4a)

\[
a_{12} \equiv \frac{\partial \dot{K}_s}{\partial I_s} = K_s,
\]  

(I.4b)

\[
a_{21} \equiv \frac{\partial \dot{I}_s}{\partial K_s} = -\frac{1}{C''(I_s)}(1 - \tau_s) \left[ F_{KK} + F_{KL} \frac{dL^m}{dK_s} \right],
\]  

(I.4c)

\[
a_{22} \equiv \frac{\partial \dot{I}_s}{\partial I_s} = r - I_s,
\]  

(I.4d)
Suppose that the economy is initially near its steady state, such that $K_s \approx \tilde{K}$ and $I_s \approx 0$. Suppose furthermore that the MNE considers the tax rate to be near its steady state, such that $\tau_s = \tau_t$ for all $s \geq t$. Evaluating all terms in matrix $J$ under these conditions and using Equation (11) to substitute for $d\ell_m^m/dK_s$, we get

$$
\begin{pmatrix}
\dot{K}_s \\
\dot{I}_s
\end{pmatrix}
\approx
\begin{pmatrix}
0 & \tilde{K} \\
-(1-\tau_t)F_{KK}G_{LL} & r
\end{pmatrix}
\begin{pmatrix}
K_s - \tilde{K} \\
I_s
\end{pmatrix},
$$

(I.5)

(I.5) is a system of two linear differential equations. The particular solution to this system is $K_s = \tilde{K}$ and $I_s = 0$. The solution to the homogeneous part is found by the derivation of the eigenvalues of the matrix $J$, $\hat{\mu}_i$, which are the roots of the characteristic equation

$$
-\hat{\mu}_i (r - \hat{\mu}_i) + \frac{(1-\tau_t)F_{KK}G_{LL}\tilde{K}}{C''(0)(F_{LL} + G_{LL})} = 0.
$$

(I.6)

Equation (I.6) has two solutions $i = 1, 2$, given by

$$
\hat{\mu}_1 = \frac{r - \sqrt{r^2 - \frac{4(1-\tau_t)F_{KK}G_{LL}\tilde{K}}{C''(F_{LL} + G_{LL})}}}{2} < 0,
\hat{\mu}_2 = \frac{r + \sqrt{r^2 - \frac{4(1-\tau_t)F_{KK}G_{LL}\tilde{K}}{C''(F_{LL} + G_{LL})}}}{2} > 0.
$$

(I.7)

Thus, there is one positive and one negative eigenvalue. The solution to the homogeneous equation is, thus,

$$
K_s = A_1 e^{\hat{\mu}_1(s-t)} + A_2 e^{\hat{\mu}_2(s-t)},
$$

$I_s = \frac{\hat{\mu}_1}{\tilde{K}}(K_s - \tilde{K})$,

(I.8)

(I.9)

where $A_1$ and $A_2$ are undetermined coefficients. The general solution is the sum of the homogeneous and particular solutions:

$$
K_s = \tilde{K} + A_1 e^{\hat{\mu}_1(s-t)} + A_2 e^{\hat{\mu}_2(s-t)},
$$

$I_s = \frac{\hat{\mu}_1}{\tilde{K}}(K_s - \tilde{K})$,

(I.10)

(I.11)

Invoking the initial condition $K(t) = K_t$ and the terminal condition $\lim_{s \to \infty} K_s = \tilde{K}$, we get $A_1 = (K_t - \tilde{K})$ and $A_2 = 0$. Thus, Equations (I.10) and (I.11) give Equations (26) and (27) in Lemma 3.

Next, I derive implicitly $I_t \equiv \iota(K_t, \tau_t)$. Using Equation (I.11), we can derive

$$
\frac{dI_t}{dK_t} = \frac{\hat{\mu}_1}{K} < 0.
$$

(I.12)
According to (I.12), a higher capital stock in period $t$, holding the steady state capital stock $\bar{K}$ unchanged, is associated with lower investment in the same period, as less investment is required to reach the steady state.

One can now use (11) and Lemma 3 to derive the impact of the tax in period $t$ on investment $I_t$. Note first that the capital stock in period $t$ is predetermined from the previous periods and cannot be affected by a change in the tax rate. Hence, a disturbance $d\tau_t$ leaves $K_t$ unchanged. Moreover, according to Equations (2), (8) and (9), $L_t^m$, $L_t^d$ and $w_t$ are also unaffected. However, the firm can change investment. Differentiate totally Equation (I.2) with respect to $\tau_t$, $I_t$ and $\dot{I}_t$ to get

$$C''(I_t)d\dot{I}_t + C'''(I_t)dI_t = (F(K_t, L_t^m) - rb)d\tau_t + C''(I_t)(r - I_t)dI_t. \quad (I.13)$$

By differentiating (I.11) with respect to time, one derives $\dot{I}_t = \ddot{\mu}_1 I_t$. Evaluate Equation (I.13) around steady state where $I_t = \dot{I}_t \approx 0$ and $d\dot{I}_t = \ddot{\mu}_1 dI_t$. The resulting expression is

$$dI_t d\tau_t \equiv \frac{dI_t}{d\tau_t} = \frac{F_K(K_t, L_t^m) - rb}{C''(0)(\ddot{\mu}_1 - r)} = -\frac{\ddot{\mu}_1(F_K(K_t, L_t^m) - rb)(F_{LL} + G_{LL})}{(1 - \tau_t)F_{KK}G_{LL}\bar{K}} < 0, \quad (I.14)$$

where I used Equation (I.6) to substitute for $r - \ddot{\mu}_1$ in the first row of (I.14).

### J Proof of Proposition 4

Define the Hamiltonian of the government as

$$H_t = \tau_t[F(K_t, \ell_t^m(K_t)) - rbK_t] + (1 - \tau_t)\omega_t(K_t)\ell_t^m(K_t) + G(D, \ell_t^d(K_t)) - (1 - \tau_t)(1 - \beta)[G(D, \ell_t^d(K_t)) - \omega_t(K_t)\ell_t^d(K_t)] + \nu_t t_t(K_t)K_t, \quad (J.1)$$

where $\nu_t$ is the co-state variable. The first-order conditions are

$$\frac{\partial H_t}{\partial \tau_t} = F(K_t, L_t^m) - brK_t - wL_t^m + (1 - \beta)[G(D, L_t^d) - \omega_t(K_t)L_t^d] + \nu_t K_t \frac{dI_t}{d\tau_t} = 0, \quad (J.2)$$

$$\nu_t r - \dot{\nu}_t = \tau_t(F_K(K_t, L_t^m) - rb) + (1 - \tau_t)(L_t^m + (1 - \beta)L_t^d) \frac{d\omega_t}{dK_t} + \nu_t \left[ I_t + K_t \frac{dI_t}{dK_t} \right]. \quad (J.3)$$
I derive first the steady state tax rate $\bar{\tau}$. Suppose that the economy is in steady state with $I_t = \bar{I} = 0$, $K_t = \bar{K}$, $\tau_t = \bar{\tau}$, $L_t^i = \bar{L}^i$, $i = m, d$. Then, $d\nu_t/d\tau_t$ becomes

$$\frac{d\nu_t}{d\tau_t}(\tau_t = \bar{\tau}, K_t = \bar{K}) = -\frac{\tilde{\mu}_1(F_K(\bar{K}, \bar{L}^m) - r b)(F_{LL} + G_{LL})}{(1 - \bar{\tau})F_{KK}G_{LL}\bar{K}} < 0, \quad (J.4)$$

where $\tilde{\mu}_1$ is the value of the rate of convergence for $\tau_t = \bar{\tau}$. In the subsequent analysis, I will use $\hat{\mu}_1$ to denote $\tilde{\mu}_1$ whenever no ambiguity arises.

To simplify notation, express Equation (J.4) as

$$\frac{d\nu_t}{d\tau_t}(\tau_t = \bar{\tau}, K_t = \bar{K}) = \frac{\hat{\mu}_1 d\bar{K}}{K d\tau} < 0, \quad (J.5)$$

where $d\bar{K}/d\tau$ is defined in Equation (C.1). Next, evaluate Equation (J.3) in steady state, which is characterized by $\dot{\tau}$, the unique solution to (J.8) is

$$\dot{\bar{\nu}} = \frac{1}{r - \hat{\mu}_1} \left[ \bar{\tau}(F_K(\bar{K}, \bar{L}^m) - r b) + (1 - \bar{\tau})(\bar{L}^m + (1 - \beta)\bar{L}^d) \frac{d\bar{\omega}}{d\bar{K}} \right]. \quad (J.6)$$

Evaluate now (J.2) in steady state and substitute $\bar{\nu}$ using Equation (J.6) to get

$$F(\bar{K}, \bar{L}^m) - r b \bar{K} - \bar{w} \bar{L}^m + (1 - \beta)[G(D, \bar{L}^d) - \bar{w} \bar{L}^d]$$

$$- \frac{\hat{\mu}_1}{r - \hat{\mu}_1} \left[ \bar{\tau}(F_K(\bar{K}, \bar{L}^m) - r b) + (1 - \bar{\tau})(\bar{L}^m + (1 - \beta)\bar{L}^d) \frac{d\bar{\omega}}{d\bar{K}} \right] \frac{d\bar{K}}{d\tau} = 0. \quad (J.7)$$

Define the effect of a change in the steady state tax rate on the steady state wage as $d\omega/d\tau$ as in Equation (D.5) and simplify $(1 - \bar{\tau})\bar{L}^m d\omega/d\tau$ analogously to (D.8). Then, (J.7) can be rewritten as

$$(F_K(\bar{K}, \bar{L}^m) - r b)\bar{K} + (1 - \beta)[G(D, \bar{L}^d) - \bar{w} \bar{L}^d] - \frac{\hat{\mu}_1}{r - \hat{\mu}_1} \bar{\tau}(F_K(\bar{K}, \bar{L}^m) - r b) \frac{d\bar{K}}{d\tau}$$

$$+ \frac{\hat{\mu}_1}{r - \hat{\mu}_1} (F_K(\bar{K}, \bar{L}^m) - r b)\bar{K} - \frac{\hat{\mu}_1}{r - \hat{\mu}_1} (1 - \bar{\tau})(1 - \beta)\bar{L}^d \frac{d\bar{\omega}}{d\tau} = 0. \quad (J.8)$$

Rearrangement of (J.8) gives Equation (18) in the main text. Since $\hat{\mu}_1$ depends on $\bar{\tau}$ in steady state, the unique solution to (J.8) is $\bar{\tau} = \bar{\tau}^*$.

Next, I derive the optimal trajectory of the tax rate. The first step is to differentiate the first-order condition with respect to $\tau_t$ (J.2) with respect to time. This gives

$$\nu_t K_t \frac{d^2\nu_t}{d\tau_t^2} \hat{\tau}_t + K_t \frac{d\nu_t}{d\tau_t} \hat{\nu}_t + \left[ F_K - r b - \frac{d\omega_t}{dK_t}(L_t^m + (1 - \beta)L_t^d) + \nu_t \frac{d\nu_t}{d\tau_t} + \nu_t K_t \frac{d^2\nu_t}{d\tau_t dK_t} \right] \hat{K}_t = 0.$$
The second derivatives of investment $\iota_t$ with respect to the tax rate and the capital stock can be derived using Equation (J.4)

\[
\frac{d^2\iota_t}{d\tau_t^2} = -\frac{\hat{\mu}_1}{(r - 2\hat{\mu}_1)(1 - \tau_t)} \frac{d\iota_t}{d\tau_t}, \quad (J.10)
\]
\[
\frac{d^2\iota_t}{d\tau_t dK_t} = -\frac{\hat{\mu}_1}{(1 - \tau_t) \bar{K}}. \quad (J.11)
\]

Solve next Equation (J.3) for $\dot{\tau}_t$ and insert it together with (J.10) and (J.11) in (J.9), we get

\[
0 = -\frac{\hat{\mu}_1 \nu_t K_t}{(r - 2\hat{\mu}_1)(1 - \tau_t)} \frac{d\iota_t}{d\tau_t} \dot{\tau}_t \quad (J.12)
\]
\[
+ \left[ \nu_t \left( r - I_t - K_t \frac{d\iota_t}{dK_t} \right) - \tau_t (F_K - rb) - (1 - \tau_t) \frac{d\omega_t}{dK_t} (L_t^m + (1 - \beta) L_t^d) \right] K_t \frac{d\iota_t}{d\tau_t}
\]
\[
+ \left[ F_K - rb - \frac{d\omega_t}{dK_t} (L_t^m + (1 - \beta) L_t^d) + \nu_t \frac{d\iota_t}{d\tau_t} - \nu_t K_t \frac{\hat{\mu}_1}{(1 - \tau_t) \bar{K}} \right] \dot{K}_t.
\]

Use now $\dot{K}_t = I_t K_t$ and $d\iota_t/dK_t = \hat{\mu}_1/\bar{K}$ from Equations (4) and (J.5), respectively, and solve (J.12) for $\dot{\tau}_t$:

\[
\dot{\tau}_t = \frac{(r - 2\hat{\mu}_1)(1 - \tau_t)}{\hat{\mu}_1 \nu_t} \left\{ \nu_t \left( r - K_t \frac{\hat{\mu}_1}{\bar{K}} \right) - \tau_t (F_K - rb) - (1 - \tau_t) \frac{d\omega_t}{dK_t} (L_t^m + (1 - \beta) L_t^d) \right\}
\]
\[
+ \left[ F_K - rb - \frac{d\omega_t}{dK_t} (L_t^m + (1 - \beta) L_t^d) - \nu_t K_t \frac{\hat{\mu}_1}{(1 - \tau_t) \bar{K}} \right] \frac{I_t}{d\iota_t/d\tau_t} \right\}. \quad (J.13)
\]

Equation (J.13) defines the change in the optimal tax rate over time as a function of $\tau_t, \nu_t, K_t$ and $I_t$. Linearize (J.13) around steady state. To do so, denote the right-hand side of (J.13) as $\Delta$. Then, $\hat{\tau}_t$ is approximately equal to

\[
\hat{\tau}_t \approx \frac{\partial \Delta}{\partial \tau} (\tau_t - \bar{\tau}) + \frac{\partial \Delta}{\partial \nu} (\nu_t - \bar{\nu}) + \frac{\partial \Delta}{\partial K} (K_t - \bar{K}) + \frac{\partial \Delta}{\partial I} (I_t - \bar{I}), \quad (J.14)
\]

where we evaluate the derivatives of $\Delta$ at the steady state, where $\hat{\tau} = \bar{K} = \bar{\nu} = \bar{I} = 0$ and $K_t = \bar{K}, \tau_t = \bar{\tau}, \nu_t = \bar{\nu}, I_t = \bar{I}$. The approximation of (J.13) leads to the following expression:

\[
\hat{\tau}_t \approx -(r - 2\hat{\mu}_1)(1 - \bar{\tau}) \left\{ \bar{\nu} \frac{d\hat{\mu}_1}{d\tau} + (F_K - rb) - \frac{d\omega_t}{dK} (\bar{L}_m + (1 - \beta) \bar{L}_d) \right\} (\tau_t - \bar{\tau})
\]

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Second, we linearize it around the steady state to get
\[
-(r - \hat{\mu}_1)(\nu_t - \bar{\nu}) + \left[ \frac{\bar{\nu}\hat{\mu}_1}{K} + \bar{\tau} \frac{F_{KK}G_{LL}}{F_{LL} + G_{LL}} + (1 - \bar{\tau})\beta \frac{d\bar{\omega}}{dK} \frac{d\bar{m}}{dK} \right] (K_t - \tilde{K})
\]
\[
+ \left[ F_K - rb - \frac{d\bar{\omega}}{dK} (\bar{L}^m + (1 - \beta)\bar{L}^d) - \frac{\bar{\nu}\hat{\mu}_1}{(1 - \bar{\tau})} \right] \left( \frac{I_t - \tilde{I}}{d\mu_t/d\tau_t} \right),
\]
where \( \hat{\mu}_1 \) is used to denote \( \tilde{\mu}_1 \). Use now Equations (I.11) and (J.5) to express \( I_t \) as \( \tilde{\mu}_1(K_t - \tilde{K})/\tilde{K} \) and \( d\mu_t/d\tau_t \) as \( -\tilde{\mu}_1/\tilde{K}(d\tilde{K}/d\tau) \). Thus, (J.15) can be simplified to
\[
\dot{\tau}_t \approx -\frac{(r - 2\hat{\mu}_1)(1 - \bar{\tau})}{\hat{\mu}_1\bar{\nu}} \left\{ \left[ \frac{\bar{\nu}\hat{\mu}_1}{K} + \bar{\tau} \frac{F_{KK}G_{LL}}{F_{LL} + G_{LL}} + (1 - \bar{\tau})\beta \frac{d\bar{\omega}}{dK} \frac{d\bar{m}}{dK} \right] (\tau_t - \bar{\tau})
\]
\[-(r - \hat{\mu}_1)(\nu_t - \bar{\nu}) + \left[ \frac{\bar{\nu}\hat{\mu}_1}{K} + \bar{\tau} \frac{F_{KK}G_{LL}}{F_{LL} + G_{LL}} + (1 - \bar{\tau})\beta \frac{d\bar{\omega}}{dK} \frac{d\bar{m}}{dK} \right]
\]
\[
+ \frac{F_K - rb - \frac{d\bar{\omega}}{dK} (\bar{L}^m + (1 - \beta)\bar{L}^d) - \frac{\bar{\nu}\hat{\mu}_1}{(1 - \bar{\tau})} \frac{1}{dK/d\tau} \right] (K_t - \tilde{K}) \right\}. \tag{J.16}
\]
Equation (J.16) expresses the change in the optimal tax rate as a function of \( \tau_t, \nu_t \) and \( K_t \). This equation forms a nonhomogeneous system of two differential equations together with Equation (J.3). To solve the system, we use the same method to linearize Equation (J.3) around steady state. First, we rewrite (J.3):
\[
\dot{\nu}_t = \nu_t \left[ r - I_t - \frac{\hat{\mu}_1 K_t}{K} \right] - \tau_t (F_K(K_t, L^m_t) - rb) - (1 - \tau_t) \frac{d\omega_t}{dK_t} (L^m_t + (1 - \beta)L^d_t). \tag{J.17}
\]
Second, we linearize it around the steady state to get
\[
\dot{\nu}_t \approx (r - \hat{\mu}_1)(\nu_t - \bar{\nu}) - \left[ \frac{\bar{\nu}\hat{\mu}_1}{K} + \bar{\tau} \frac{F_{KK}G_{LL}}{F_{LL} + G_{LL}} + (1 - \bar{\tau})\beta \frac{d\bar{\omega}}{dK} \frac{d\bar{m}}{dK} \right] (\tau_t - \bar{\tau})
\]
\[- \left[ \frac{2\bar{\nu}\hat{\mu}_1}{K} + \bar{\tau} \frac{F_{KK}G_{LL}}{F_{LL} + G_{LL}} + (1 - \bar{\tau})\beta \frac{d\bar{\omega}}{dK} \frac{d\bar{m}}{dK} \right] (K_t - \tilde{K}). \tag{J.18}
\]
Equations (J.16) and (J.18) together form a system of two linear differential equations that can be solved for \( \tau_t, \nu_t \) as functions of capital \( K_t \). To solve them, first we rewrite the system in matrix form:
\[
\begin{pmatrix} \dot{\nu}_t \\ \dot{\tau}_t \end{pmatrix} \approx \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \nu_t - \bar{\nu} \\ \tau_t - \bar{\tau} \end{pmatrix} + \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} (K_t - \tilde{K}). \tag{J.19}
\]
where the constant terms $a_{ij}, i, j = 1, \ldots, 3$ are defined in (J.16) and (J.19) as

\begin{align*}
a_{11} & \equiv r - \hat{\mu}_1, \\
a_{12} & \equiv -\left[\frac{\tilde{\nu}}{K} \frac{d\hat{\mu}_1}{d\tau} + (F_K - rb) - \frac{d\bar{\omega}}{dK}(\bar{\omega}^m + (1 - \beta)\ell^d)\right], \\
a_{13} & \equiv -\left[\frac{2\tilde{\nu}\hat{\mu}_1}{K} + \tilde{\tau} \frac{F_{KK} G_{LL}}{F_{LL} + G_{LL}} + (1 - \tilde{\tau})\beta \frac{d\bar{\omega}}{dK} \frac{d\bar{m}}{d\ell^m}\right], \\
a_{21} & \equiv \frac{(r - 2\hat{\mu}_1)(1 - \tilde{\tau})}{\hat{\mu}_1\tilde{\nu}} a_{11}, \\
a_{22} & \equiv \frac{(r - 2\hat{\mu}_1)(1 - \tilde{\tau})}{\hat{\mu}_1\tilde{\nu}} a_{12}, \\
a_{23} & \equiv -(r - 2\hat{\mu}_1)(1 - \tilde{\tau}) \left[\frac{\tilde{\nu}}{K} \frac{d\hat{\mu}_1}{d\tau} + \tilde{\tau} \frac{F_{KK} G_{LL}}{F_{LL} + G_{LL}} + (1 - \tilde{\tau})\beta \frac{d\bar{\omega}}{dK} \frac{d\bar{m}}{d\ell^m}\right]
\quad + \left[F_K - rb - \frac{d\bar{\omega}}{dK}(\bar{\omega}^m + (1 - \beta)\ell^d) - \frac{\tilde{\nu}}{(1 - \tilde{\tau})} \frac{1}{dK/d\tau}\right].
\end{align*}

The system (J.19) is nonhomogeneous. Its general solution is a sum of the solution to the homogeneous part (involving the matrix $J$) and the solution to the nonhomogeneous part (involving the evolution of the capital stock $K_t - \bar{K}$). To solve the homogeneous part, note that the columns of the matrix $J$ are not linearly independent and its determinant equals zero. Therefore, the homogeneous part of the system (J.19) can be written as

\begin{equation*}
\dot{\tau}_t = \frac{(r - 2\hat{\mu}_1)(1 - \tilde{\tau})}{\hat{\mu}_1\tilde{\nu}} [a_{11}(\tau_t - \tilde{\tau}) + a_{12}(\nu_t - \tilde{\nu})] = \frac{(r - 2\hat{\mu}_1)(1 - \tilde{\tau})}{\hat{\mu}_1\tilde{\nu}} \nu_t. \tag{J.21}
\end{equation*}

This is a separable differential equation. Its solution is

\begin{equation*}
\tau_t - \tilde{\tau} = \frac{(r - 2\hat{\mu}_1)(1 - \tilde{\tau})}{\hat{\mu}_1\tilde{\nu}} (\nu_t - \tilde{\nu}). \tag{J.22}
\end{equation*}

Inserting (J.22) in (J.21), we get

\begin{equation*}
\dot{\nu}_t = \left[a_{11} + \frac{(r - 2\hat{\mu}_1)(1 - \tilde{\tau})}{\hat{\mu}_1\tilde{\nu}} a_{12}\right] (\nu_t - \tilde{\nu}) \equiv A(\nu_t - \tilde{\nu}). \tag{J.23}
\end{equation*}

Equation (J.23) is a differential equation with the solution $\nu_t = \tilde{\nu} + e^A(\nu_0 - \tilde{\nu})$ if $A < 0$ and $\nu_t = \tilde{\nu}$ for all $t$ if $A > 0$. The reason for the second result is that a positive constant $A$ results in unbounded growth which is incompatible with steady state. To find the
sign of $A$, I simplify it:

$$A = a_{11} + \frac{(r - 2\hat{\mu}_1)(1 - \tilde{\tau})}{\hat{\mu}_1 \tilde{\nu}} a_{12}$$

$$= (r - \hat{\mu}_1) - \frac{(r - 2\hat{\mu}_1)(1 - \tilde{\tau})}{\hat{\mu}_1 \tilde{\nu}} \left[ \frac{\tilde{\nu} d\hat{\mu}_1}{K} + (F_K - rb) - \frac{d\bar{\omega}}{dK} (\tilde{L}^m + (1 - \beta)\tilde{L}^d) \right]$$

$$= - \frac{(F_K - rb)(r(1 - \tilde{\tau}) - 2\hat{\mu}_1) - r(1 - \tilde{\tau}) \frac{d\bar{\omega}}{dK} (\tilde{L}^m + (1 - \beta)\tilde{L}^d)}{\hat{\mu}_1 \tilde{\nu}}. \quad (J.24)$$

Note that the last term of $(J.24)$ is proportional to the second derivative of the Hamiltonian with respect to the tax rate, i.e. the derivative of $(J.2)$ with respect to $\tau$. The second-order condition requires it to be negative. Thus, $A > 0$ and the solution to $(J.24)$ is $\nu_t^h = \tilde{\nu}$ for all $t$, where the superscript $h$ stays for homogeneous. However, this also means that the solution to the homogenous part also involves $\tau_t^h = \tilde{\tau}$ for all $t$.

The nonhomogeneous part of $(J.19)$ gives $\dot{\nu}_t$ and $\dot{\tau}_t$ as functions of the capital stock $K_t - \tilde{K}$. Lemma 3 states that

$$K_t - \tilde{K} = (K_0 - \tilde{K})e^{\hat{\mu}_1t}, \quad (J.25)$$

where $\hat{\mu}_1$ is a function of $\tau_0$. Therefore, we look for a particular solution to the nonhomogeneous part of the system $(J.19)$ for the tax rate of the form $\tau_t^{nh} = Be^{\hat{\mu}_1t}$ for some constant $B$ (the superscript $nh$ stays for nonhomogeneous). Thus, the general solution becomes $\tau_t = \tau_t^h + \tau_t^{nh} = \tilde{\tau} + \tau_t^{nh}$. Differentiate this candidate for general solution with respect to time:

$$\dot{\tau}_t = \dot{\tau}_t^{nh} = \hat{\mu}_1(\tau_0)Be^{\hat{\mu}_1t}. \quad (J.26)$$

However, the system $(J.19)$ states that $\dot{\tau}_t = a_{23}(K_0 - \tilde{K})e^{\hat{\mu}_1t}$. Equating this expression and $(J.26)$, one can solve for $B$:

$$B = \frac{a_{23}(K_0 - \tilde{K})}{\hat{\mu}_1}. \quad (J.27)$$

Thus, the general solution for $\tau_t$ becomes

$$\tau_t = \tilde{\tau} + \frac{a_{23}}{\hat{\mu}_1}(K_0 - \tilde{K})e^{\hat{\mu}_1t}. \quad (J.28)$$

It remains to prove that $a_{23} < 0$, where $a_{23}$ is defined in $(J.20f)$. After some manipula-
tion, one can show that (J.20f) can be rewritten as

\[ a_{23} = -\frac{(r-2\bar{\mu}_1)(1-\bar{\tau})}{\bar{\mu}_1 \bar{\nu} \frac{d\bar{K}}{d\tau}} \left[ \frac{d\bar{K}}{d\tau} \frac{\bar{\nu} \bar{\mu}_1}{\bar{K}} + \frac{d\bar{K}}{d\tau} \frac{F_{KK}G_{LL}}{F_{LL} + G_{LL}} + \frac{d\bar{K}}{d\tau} (1-\bar{\tau}) \beta \frac{d\bar{\omega}}{d\bar{K}} \frac{d\bar{\omega}}{d\bar{K}} \right] \\
+ \frac{F_K - rb}{\bar{\mu}_1 \bar{\nu} \frac{d\bar{K}}{d\tau}} \left[ \frac{d\bar{K}}{d\tau} \frac{\bar{\nu} \bar{\mu}_1}{\bar{K}} + \frac{d\bar{K}}{d\tau} \frac{F_{KK}G_{LL}}{F_{LL} + G_{LL}} + \frac{d\bar{K}}{d\tau} (1-\bar{\tau}) \beta \frac{d\bar{\omega}}{d\bar{K}} \frac{d\bar{\omega}}{d\bar{K}} \right] \\
+ (F_K - rb) \frac{(r-\bar{\mu}_1)(1-\bar{\tau}) - \bar{\mu}_1 \bar{\tau}}{(r-\bar{\mu}_1)(1-\bar{\tau})} \frac{r}{r-\bar{\mu}_1} \frac{d\bar{\omega}}{d\bar{K}} (\bar{L}^m + (1-\beta)\bar{L}^d) \right). \] (J.29)

where to derive the last expression, I inserted the value of \( \bar{\nu} \) from Equation (J.6) in the second row of \( a_{23} \). Substitute now for \( d\bar{K}/d\tau, d\bar{\omega}/d\bar{K} \) and \( d\bar{\omega}/d\bar{K} \) in the second and third term in brackets in the first row of (J.29) and use the constant returns property \( F_{KL}^2 = F_{KK}F_{LL} \) to derive

\[ a_{23} = -\frac{(r-2\bar{\mu}_1)(1-\bar{\tau})}{\bar{\mu}_1 \bar{\nu} \frac{d\bar{K}}{d\tau}} \left[ \frac{d\bar{K}}{d\tau} \frac{\bar{\nu} \bar{\mu}_1}{\bar{K}} + \frac{d\bar{K}}{d\tau} \frac{F_{KK}G_{LL}}{F_{LL} + G_{LL}} + \frac{d\bar{K}}{d\tau} (1-\bar{\tau}) \beta \frac{F_{LL}}{F_{LL} + G_{LL}} \right] \\
+ \frac{F_K - rb}{\bar{\mu}_1 \bar{\nu} \frac{d\bar{K}}{d\tau}} \left[ \frac{d\bar{K}}{d\tau} \frac{\bar{\nu} \bar{\mu}_1}{\bar{K}} + \frac{d\bar{K}}{d\tau} \frac{F_{KK}G_{LL}}{F_{LL} + G_{LL}} + \frac{d\bar{K}}{d\tau} (1-\bar{\tau}) \beta \frac{F_{LL}}{F_{LL} + G_{LL}} \right] \\
+ (F_K - rb) \frac{(r-\bar{\mu}_1)(1-\bar{\tau}) - \bar{\mu}_1 \bar{\tau}}{(r-\bar{\mu}_1)(1-\bar{\tau})} \frac{r}{r-\bar{\mu}_1} \frac{d\bar{\omega}}{d\bar{K}} (\bar{L}^m + (1-\beta)\bar{L}^d) \right]. \] (J.30)

The first term in brackets can be substituted from Equation (J.2), when evaluated in steady state, which gives

\[ \frac{\mu_1 \bar{\nu} \frac{d\bar{K}}{d\tau}}{d\tau} = F_{KL}^2 = F_{KK}F_{LL} \]

\[ = F(\bar{K}, \bar{L}^m) - br\bar{K} - \bar{\omega}\bar{L}^m + (1-\beta)[G(D, \bar{L}^d) - \bar{w}\bar{L}^d] \]

\[ = (F_K - rb)\bar{K} + (1-\beta)[G(D, \bar{L}^d) - \bar{w}\bar{L}^d]. \] (J.31)

In rewriting (J.2) above, I used the constant returns property \( F_{KL}^2 = F_{KK}F_{LL} = F \). Use (J.31) to substitute for the first term in brackets in (J.30). After some calculations, one gets

\[ a_{23} = -\frac{(r-2\bar{\mu}_1)(1-\bar{\tau})}{\bar{\mu}_1 \bar{\nu} \frac{d\bar{K}}{d\tau}} \left[ (1-\beta)[G(D, \bar{L}^d) - \bar{w}\bar{L}^d] + (F_K - rb) \left( 1 - \beta \frac{F_{LL}}{F_{LL} + G_{LL}} \right) \right] \\
+ (F_K - rb) \frac{r}{r-\bar{\mu}_1} \frac{d\bar{\omega}}{d\bar{K}} (\bar{L}^m + (1-\beta)\bar{L}^d) \right]. \] (J.32)

All the terms in brackets are positive, except for the last one on the second row. Furthermore, the brackets are multiplied by a negative term. Therefore, \( a_{23} \) is negative if the terms in brackets are positive, i.e., if the last term in brackets is not too negative.
However, the second-order condition (F.7) (which must also be satisfied in the case of a time-varying tax rate in steady state) gives us an upper bound for this term. Using (F.7) and performing some tedious calculations, one can show that

\[
a_{23} \leq -\frac{(r - 2\mu_1)(1 - \bar{\tau})}{\bar{\mu}_1} \left[ (1 - \beta)[G(D, \tilde{L}^d) - \bar{w}L^d] + \right. \\
+ \frac{(F_K - rb)}{(r - 3\mu_1)(r - \bar{\mu}_1)(1 - \bar{\tau})} \left[ ((r - \bar{\mu}_1)^2 - \bar{\mu}_1r)(1 - \bar{\tau}) \left( 1 - \beta \frac{F_{LL}}{F_{LL} + G_{LL}} \right) \right. \\
+ \bar{\tau}r^2 - \bar{\mu}_1r(1 + 2\bar{\tau}) + \hat{\mu}_1^2(1 - \bar{\tau}) \right] \\
+ \frac{\bar{\mu}_1r(1 - \bar{\tau})}{(r - 3\mu_1)(r - \bar{\mu}_1)K} d\tilde{K} \left[ \left( \frac{F_K - rb}{1 - \bar{\tau}} \right) + \frac{d\bar{\omega}}{d\tilde{K}} (\tilde{L}^m + (1 - \beta)\tilde{L}^d) \right] \left. \right] < 0.
\]

Hence, \( a_{23} \) has an upper bound that is negative. \( \square \)

**K  Proof of Proposition 5**

Proposition 5 states that Proposition 1 holds in the case of a time-varying tax rate when one replaces \( \tau^* \) by \( \bar{\tau} \). To prove this, note that \( \tau^* \) and \( \bar{\tau} \) coincide (see Proposition 4). Moreover, \( \tilde{K} \) is determined by (10a) in both situations. Hence, Proposition 1 can be proven again using Equations (10a) and (18).

Second, Proposition 5 states that Propositions 2 is qualitatively unchanged. Because \( \tau^* = \bar{\tau} \), all long-term effects of a change in internal debt \( db \) remains exactly the same as derived in Proposition 2. Moreover, if the economy is static \( (\mu_1 \rightarrow \infty) \), the time-varying tax rate model collapses to a static model with a constant tax rate. Therefore, it remains to prove that Equation (23) holds. To derive the initial impact of \( b \) on welfare, differentiate Equation (G.12) with respect to \( b \):

\[
\frac{d\Omega_0}{db} = -\tau_0 r \tilde{K}_0 + \left[ F(\tilde{K}_0, \tilde{L}_0^m) - rb\tilde{K}_0 - \bar{w}L_0^m + (1 - \beta)(G(D, \tilde{L}_0^d) - \bar{w}\tilde{L}_0^d) \right] \frac{d\tau_0}{db}.
\]

(K.1)

Evaluate (K.1) at \( \beta = 1 \) and use the the constant returns property \( F = F_K K + F_L L^m \):

\[
\frac{d\Omega_0}{db} (\beta = 1) = -\tau_0 r \tilde{K}_0 + \left( F_K(\tilde{K}_0, \tilde{L}_0^m) - rb \right) \tilde{K}_0 \frac{d\tau_0}{db}.
\]

(K.2)

The change in the initial tax rate is, according to Equation (32):

\[
\frac{d\tau_0}{db} = \frac{d\bar{\tau}}{db} - \frac{a_{23}}{\bar{\mu}_1} \frac{d\tilde{K}}{db} - \frac{a_{23}}{\bar{\mu}_1^2} \frac{d\tilde{\mu}_1}{d\tau_0} \frac{d\tilde{K}}{db}(K_0 - \tilde{K})
\]

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\[\frac{d\tau_0}{db} = \frac{1}{\Gamma} \left[ \frac{d\tilde{\tau}}{db} - \frac{a_{23} d\tilde{K}}{\mu_1 \frac{\partial}{\partial b}} \right], \quad (K.3)\]

where

\[\Gamma = 1 + \frac{a_{23} \frac{d\tilde{\mu}_1}{\partial b}}{\mu_1^2} (K_0 - \tilde{K}). \quad (K.4)\]

Note that, starting from a steady state, \(K_0 - \tilde{K} = \tilde{K}_0 - \tilde{K}_i\), where \(\tilde{K}_i\) is the steady state capital stock associated with internal debt \(b_i, i = 0, 1\). Thus, \(K_0 - \tilde{K} \approx -\frac{d\tilde{K}}{db} db\). Because \(d\tilde{K}/db\) is different from zero, while the change \(db\) is infinitesimal, the second term in (K.4) is also infinitesimal and \(\Gamma \approx 1\).

When the change in internal debt \(db\) is small, then \(\tilde{K}_0 \approx \tilde{K}\) and \(\tilde{L}_m^0 \approx \tilde{L}_m, \tau_0 \approx \tilde{\tau}\). Hence, Equation (K.2) can be evaluated in the vicinity of \(\tilde{K}, \tilde{\tau}\), which gives

\[\frac{d\Omega_0}{db} (\beta = 1) = -\tilde{\tau} r \tilde{K} + \left( F_K(\tilde{K}, \tilde{L}_m) - rb \right) \tilde{K} \left[ \frac{d\tilde{\tau}}{db} - \frac{a_{23} d\tilde{K}}{\mu_1 \frac{\partial}{\partial b}} \right] \]

\[= -\tilde{K} \left[ r\tilde{\tau} - (F_K - rb) \frac{d\tilde{\tau}}{db} \right] - (F_K - rb) \tilde{K} \frac{a_{23} \frac{d\tilde{K}}{db}}{\mu_1} \]

\[= \frac{\mu_1 d\tilde{\Omega}}{r \frac{\partial}{\partial b}} - (F_K - rb) \tilde{K} \frac{a_{23} \frac{d\tilde{K}}{db}}{\mu_1} < 0. \quad (K.5)\]

The initial negative welfare impact, as measure by (K.5), is more pronounced than in the case of a constant tax rate. To prove that part (c) of Proposition 2 holds, note that the initial welfare impact, (K.2), is ambiguous for small values of \(\beta\).

Lastly, we prove that Proposition 3 also holds. The effect of an increase in internal debt on the overall welfare can be derived as in Appendix H. Using LaFrance and Barney (1991), the envelope theorem states

\[\frac{d}{db} \int_0^\infty \Omega_t e^{-rt} dt = \int_0^\infty \frac{\partial H_t}{\partial b} e^{-rt} dt, \quad (K.6)\]

where \(H_t\) is the Hamiltonian function. The above derivative can be simplified to

\[\int_0^\infty \left\{ -\tau_t r K_t + \left[ \tau_t (F_K - rb) + (1 - \tau_t) [L_t^m + (1 - \beta) L_t^d] \frac{d\omega_t}{dt} \right] K_t \right\} e^{-rt} dt. \quad (K.7)\]

Evaluating (K.7) around steady state with \(K_t \approx \tilde{K}\) and \(\tau_t \approx \tilde{\tau}\), it coincides with (H.4). Thus, Proposition 3 continues to hold.