Illustrating Income Mobility: New Measures*

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Abstract

Jenkins and Lambert (1997) demonstrated that a number of measures of poverty could be combined and compared using the “Three Is of Poverty” (TIP) curve; the ‘three Is’ being the incidence, intensity and inequality of poverty. This paper takes the insights from the TIP curve and applies them to income growth based measures of mobility, proposing a “Three Is of Mobility”, or TIM, curve. Similar analysis is then applied to re-ranking measures of mobility to yield re-ranking curves. Illustrations are provided using income data from random samples of New Zealand income taxpayers over the period 1998 to 2010. Both income growth and re-ranking based curves represent simple graphical devices that nevertheless conveniently illustrate the ‘three Is’ properties of income mobility.

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1 Introduction

When comparing distributions of non-negative economic variables, such as annual income or consumption, the Lorenz curve is ubiquitous. With individual observations arranged in ascending order, this plots the cumulative proportion of total income against the corresponding cumulative proportion of individuals. A normalised area measure of the distance between the Lorenz curve and the line of equality gives rise to the equally famous Gini inequality measure. Furthermore, the concept of ‘Lorenz dominance’ provides an immediate qualitative comparison between the inequality of two distributions, and this can be given a welfare interpretation when combined with the value judgement summarised by the ‘principle of transfers’. The Lorenz curve thus provides a valuable diagrammatic summary, providing much more information than either the density function or the distribution function alone.¹

Where concern is largely for those below a poverty line, an alternative diagrammatic device involves, for incomes again arranged in ascending order, plotting the cumulative poverty gap per person against the corresponding cumulative proportion of people. This is the TIP curve, named by Jenkins and Lambert (1997) for its ability to indicate the ‘Three Is of Poverty’, namely incidence (the poverty headcount), intensity (the poverty income-gap) and inequality (the within-poor distribution). For those below the poverty line, the curve is a straight line only in situations where all the poor have equal incomes. As with the Lorenz curve, dominance properties hold. For both Lorenz and TIP curves, comparisons involving intersecting curves lead to the need to impose more structure on evaluations, in the form of explicit value judgements and quantitative inequality and poverty measures.

In the context of income mobility, several diagrams have been proposed to capture the key properties of mobility in an easily-perceived way. This is complicated by the variety of definitions and interpretations of different mobility concepts. For example, following the taxonomy of mobility concepts developed by Fields (2000), and reviewed recently by Jäntti and Jenkins (2015), mobility has been characterised variously as associated with individual income growth (either absolute or relative); directional or non-directional positional change (re-ranking); income share change; impacts on the inequality of longer-term incomes; and ‘income risk'; see Fields (2000, 2008), Jäntti and Jenkins (2015).²

¹The so-called Pen Parade, following Pen (1971), is simply the distribution function rotated through 90 degrees, therefore showing income on the vertical axis and the cumulative proportion of people on the horizontal axis. It is used, along with the metaphorical parade of individuals aligned from poor to rich, mainly in popular presentations.

²Following Fields (2000) a number of authors have pointed to the normative ambiguity associated with (possibly desirable) flexibility in short-term income movements versus (undesirable) volatility. Jäntti and
Most illustrative devices for income mobility have focused on income growth measures. These include Trede (1998), Ravallion and Chen (2003), Bourguignon (2011), Van Kerm (2009) and Jenkins and Van Kerm (2006, 2011, 2016), with recent contributions emphasising the welfare dominance properties of alternative income mobility measures or illustrative devices.\(^3\) However, it is suggested below that the ‘three Is’ properties can be translated to the context of income mobility, yet none of the existing approaches focuses specifically on these three ‘positive’ properties for mobility measures analogous to the TIP curve for poverty.

This paper addresses that gap by offering new illustrative devices for income mobility. Firstly, a modification of income growth profiles is proposed to illustrate the three Is of mobility. Like Bourguignon (2011) and Jenkins and Van Kerm (2016) this captures longitudinal dimensions. With individuals ranked in ascending order of initial income, it plots the cumulative proportional income change \textit{per capita} (not per head of the cumulated subgroup) against the corresponding cumulative proportion of individuals. Since the diagram bears a close resemblance to the TIP curve it is described here as a ‘Three Is of Mobility’, or TIM curve.

Secondly, comparable devices capable of illustrating the three Is properties for a positional change measure of income mobility are developed. This first considers the cumulative observed re-ranking change across individuals ranked in ascending order of the initial income distribution and, secondly, the ratio of observed re-ranking to the maximum feasible re-ranking for each individual. The former may be called a ‘cumulative re-ranking curve’ and the latter a ‘re-ranking ratio’, or \textit{RRR}, curve.

Existing illustrative devices for income mobility are discussed in Section 2. Subsequent sections apply and illustrate the new illustrative devices for income mobility. Focusing first on the individual income growth class of measures allows the TIM curve concept to be introduced in Section 3. Section 4 then introduces positional change mobility measures and the derivation of the \textit{RRR} curve. The new devices are illustrated using a longitudinal

\(^3\)Jenkins (2015) suggest that the concept of income risk can be regarded as one aspect of longer term income inequality, where changes in an income inequality measure over time have both permanent predictable and transitory unpredictable components. They label the latter as ‘income risk’. In measuring aggregate income growth, Palmisano and Van de gaer (2016) combine individual income growth and initial rank position, to form a weighted average of individual growth with weights decreasing with initial rank. Cowell and Flachaire (2016) provide a general framework to categorise and evaluate different mobility concepts. Creedy and Wilhelm (2002) examined income mobility changes and social welfare in the context of a welfare function defined over longer-period incomes and where individuals have a preference, ceteris paribus, for stable incomes.

\(^3\)Palmisano and Peragine (2015) propose a similar welfare framework for analysing growth incidence. They argue that, unlike Bourguignon (2011) and Jenkins and Van Kerm (2011), their framework can incorporate horizontal inequality concerns.
sample of individuals from New Zealand, firstly for the TIM curve in Section 5, and for re-ranking measures in Section 6. Conclusions are in Section 7.

2 Illustrative Devices for Income Mobility

A number of authors have sought to illustrate distributional dimensions of income mobility across a population or sample of individuals. This section reviews some of the more commonly used diagrams before considering the alternatives proposed here.

2.1 Quantile Profiles

Trede (1998), motivated by a desire to illustrate and summarise the information contained in a transition matrix, concentrated on the conditional distributions of income (relative to, say, the median) in one year, given incomes in an earlier year. He proposed diagrams showing profiles of various quantiles of the conditional distributions, with relative income in the initial year on the horizontal axis. For incomes in $t$ and $t - 1$, the method involves non-parametric estimation of various quantiles of conditional distributions of $x_t$ for given values of $x_{t-1}$. The quantile profiles are shown in a diagram with income in $t$ on the vertical axis and $x_{t-1}$ on the horizontal axis. Trede suggested translating incomes to relative values by dividing by the mean or median income in each period. A simplified example is shown if Figure 1, for just three quantiles.

![Quantile Profiles Diagram]

Figure 1: Quantile Regressions of Income in $t$ on Income in $t - 1$
For ‘perfect mobility’, defined by Trede as independence of income in $t - 1$, the quantile profiles become horizontal. The vertical distances between quantile profiles give an indication of the extent of income inequality in $t$. The extreme of ‘total [relative] immobility’ produces quantile profiles that coincide: that is, all those with $x_{t-1}$ have the same relative income in period $t$. If the (common) quantile profiles coincide with the 45-degree line in the diagram, there is no change in the (marginal) distribution when moving from $t-1$ to $t$. Thus, Trede (1998, p. 80) suggests that, ‘both the distance from each other, and the slopes of the quantile lines provide information about income dynamics’. However, the marginal distributions can remain unchanged even when the quantiles do not coincide and they are not 45-degree lines; for further details, see Creedy and Gemmell (2017).

### 2.2 Growth Incidence Curves

An alternative approach, a ‘growth incidence curve’ (GIC), plots the income growth rate between two periods of each quantile or percentile of the distribution of initial incomes. As originally proposed by Ravallion and Chen (2003), the GIC is based on cross-sectional distributions for two periods and is therefore not capable of illustrating individual-specific income mobility. However, Bourguignon (2011) extends the GIC concept to capture longitudinal aspects of individual income growth in what he refers to as a ‘non-anonymous growth incidence curve’ (na-GIC), where the ‘non-anonymity’ refers to the same individuals being identified in both the initial and terminal income distributions. These na-GICs are based only on the characteristics of the two relevant (longitudinal) distributions, but can easily display relative growth differences by subtracting overall income growth.

Beginning from an initial income distribution with a distribution function given by $F(y)$, the Bourguignon (2011) na-GIC is defined over both initial and terminal period distributions by first defining a ‘quantile function’, $y_F(p)$, as the inverse of $F(y)$. A similar function $y_F(p)$, describes the equivalent terminal quantile function, conditional on initial incomes. Thus, income growth rates for each $p^{th}$ percentile are given by:

$$g_{yF}(p) = \frac{y_F(p)}{y_F(p)} - 1$$

As Bourguignon (2011, p. 609) puts it, the ‘distributional impact of growth is thus represented through the inverse of the cumulative density functions rather than those functions

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4 The vertical distances refer to conditional distributions, not the marginal distribution in $t$.

5 A similar ‘poverty growth curve’ (PGC) to the Ravallion and Chen (2003) GIC is proposed by Son (2004), who illustrates cumulative growth across $p$ percentiles of the income distribution. However, like Ravallion and Chen (2003), the PGC is based on comparisons of cross-sectional, rather than longitudinal, income distributions. See also Son (2007).
themselves’. He further proposes a cumulative version of the na-GIC, referred to as the ‘\(p\)-cumulative GIC’, given by:

\[
G_{\phi_F}(p) = \frac{\int_0^pg_\phi(q)y_F(q)\,dq}{\int_0^py_F(q)\,dq}
\]  

(2)

Bourguignon’s (2011) graphical representations involve plotting \(g_{\phi_F}(p)\) or \(G_{\phi_F}(p)\) against \(p\). His empirical applications relate to distributions of income growth rates across countries for different periods.

### 2.3 Income Growth Profiles

Jenkins and Van Kerm (2016) define Income Growth Profiles, IGPs, which are similar to those developed by Van Kerm (2009) and Bourguignon (2011); see also Grimm (2007). However, in defining IGPs much attention is addressed to the welfare dominance properties of individual income growth based on an adaptation of the Atkinson and Bourguignon (1982) social welfare function where individual utilities are based on incomes in both the initial and terminal periods; see Jenkins and Van Kerm (2016, pp. 681-3). That is, their objective is to produce summary indices of income growth with consistent welfare foundations that are helpful for normative evaluations of alternative distributions of individual income growth.

Their objective is therefore rather different from the positive measurement or description of income mobility properties pursued in the present paper. Firstly, the IGP involves plotting a measure of average income growth, \(m(p)\), for the \(p\)th percentile, against \(p\), where in their case \(m(p)\) is an expectation-based measure conditional on initial income and amenable to social welfare comparisons. The IGP clearly bears a close resemblance to the Bourguignon (2011) na-GIC but, unlike the na-GIC, it does not require a common marginal initial income distribution - helpful for evaluation of dominance properties across different samples.

Secondly, Jenkins and Van Kerm (2016) also propose a cumulative version of the IGP (a CIGP) in which a measure of average income growth for those with initial incomes below \(x(p)\) is plotted against \(p\). The CIGP is given by:

\[
C(p) = \frac{1}{p} \int_0^p m(q)\,dq
\]  

(3)

As Jenkins and Van Kerm, 2016, p. 685) say, the CIGP ‘plots areas below the income growth profile, analogous to the way that a generalized Lorenz curve shows areas below a quantile function. The slope of the cumulative income growth profile can be positive or negative at different values of \(p\).”
Figure 2: Income Growth and Cumulative Income Growth Profiles

For comparison with the diagrams suggested here, Figure 2 is taken from Jenkins and Van Kerm (2016) and shows examples of IGPs and CIGPs, obtained from various waves of the longitudinal British Household Panel Survey covering 1991 to 2006. The cumulative version on the right is conceptually closest to the TIM curves described below, with the CIGPs in this case revealing systematically lower cumulative growth at higher values of \( p \).\(^6\)

As Section 3 shows, a closely-related illustrative device to the CIGP can be deployed to illustrate similar distributional dimensions of mobility in a visually more straightforward manner.

2.4 Positional Mobility Devices

Positional mobility has been widely studied using transition matrices, which capture the movement of individuals across percentiles of the income distribution between two time periods. However, as Trede (1998) suggests, "[e]ven if the number of classes is restricted to as few as five classes the transition matrix is not easily comprehended at first sight’. Indeed this was a key motivation for Trede’s (1998) graphical contribution discussed above. In addition, the use of a small number of classes increases the likelihood that potentially important within-class mobility is obscured.

More recently, D’Agostino and Dardanoni (2009) and Cowell and Flachaire (2016) have sought to redefine and clarify various rank-related mobility concepts and measures. Cowell and Flachaire (2016) propose what they term a ‘superclass’ of rank-based mobility measures.

\(^6\)Jenkins and Van Kerm (2016) also consider income changes in absolute terms, \( dx \), as well as growth rates, \( dx/x \). Appendix B illustrates equivalent TIM curves defined over income changes, \( dx \).
They stress, for example, the importance of separating the evaluation of an individual’s positional ‘status’ from movements between positions, where measurement of the latter uses distance concepts. However, neither study considers suitable graphical devices to illustrate the measures. Section 4 therefore proposes TIM-type devices to illustrate positional mobility based on re-ranking, for which there seem to be no direct equivalents in the literature.

3 The TIM Curve

This section begins by summarising the key aspects of the TIP curve developed by Jenkins and Lambert (1997), in subsection 3.1. It is then adapted in the income mobility context in subsection 3.2.

3.1 The TIP Curve

Jenkins and Lambert (1997) demonstrated that three important dimensions of poverty can be summarised by their TIP curve. These are: the incidence of poverty, as captured by the headcount poverty measure; the intensity, as measured by the income gap, \( x_p - x_i \), where \( x_p \) is the poverty line; and the inequality of poverty within the poor group, capturing how far the incomes of the poorest differ from those closer to the threshold, \( x_p \).

Let \( x_i \) denote individual \( i \)’s income, with \( i = 1, ..., n \). Given \( x_p \), the poverty gaps are defined by \( g(x_i) = 0 \) for \( x_i > x_p \) and \( g(x_i) = x_p - x_i \) for \( x_i < x_p \). When incomes are ranked in ascending order, the TIP curve is obtained by plotting \( \frac{1}{n} \sum_{i=1}^{k} g(x_i) \) against \( \frac{k}{n} \), for \( k = 1, ..., n \). That is, the total cumulative poverty gap per capita is plotted against the associated proportion of people.

A hypothetical example is shown in Figure 3. The slope at any point is equal to the average poverty gap, with a steeper slope indicating a larger poverty gap. Flattening of the curve therefore shows the extent to which the average poverty gap falls as income rises towards \( x_p \). Thus, inequality among the poor is reflected in the curvature of the TIP curve. The curve becomes horizontal beyond \( H \), beyond which there is no one in poverty, given a prior choice of \( p \) or \( x(p) \). Poverty is unambiguously higher where a TIP curve lies wholly above and to the left of an alternative TIP curve.

The TIP curve provides useful information because, for any poverty metric (such as income or consumption) and poverty threshold definition, it captures three separate and important distributional dimensions. However, these properties relate to levels of poverty at a point in time, rather than poverty dynamics over time, such as the extent to which the poor stay poor over different time horizons. This, of course, is captured by longitudinal
income mobility analysis.

3.2 Three Is of Mobility

The three Is properties of income poverty (or other income thresholds) convey important distributional information. They can be translated to corresponding or analogous properties of income mobility, using the ‘Three Is of Mobility’ (TIM) curve, defined here. First, define the logarithm of income, \( y_i = \log x_i \), for individuals \( i = 1, ..., n \). Hence \( y_{i,t} - y_{i,t-1} \) is (approximately) person \( i \)'s proportional change in income from period \( t-1 \) to \( t \). With log incomes ranked in ascending order, plot \( \frac{1}{n} \sum_{i=1}^{k} (y_{i,t} - y_{i,t-1}) \) against \( \frac{k}{n} \), for \( k = 1, ..., n \). Thus the TIM curve plots the cumulative proportional income change per capita against the corresponding proportion of individuals.

In specifying a TIM curve, the focus is on the mobility of a particular group of low-income individuals: those with incomes below \( x(h) \), that is, for the proportion, \( h \), of the population. In this framework \( h \) captures the incidence of the particular low income group of concern, just as the headcount poverty measures the incidence of income poverty. Similarly, intensity and inequality properties of mobility can be defined. The extent, or intensity, of mobility of a particular group of individuals, such as those below \( h \) or \( x(h) \), is measured by the height of the TIM curve at the relevant point, \( h \). The inequality property of mobility refers to differences in the income growth rates within the specified group (below \( h \), and is reflected in the curvature of the relevant portion of the TIM curve. The relevance of all three
dimensions of mobility to longer-term inequality, or the persistence of poverty, suggests that a TIM curve, analogous to the TIP curve, can provide similar insights when ‘mobility as income growth’ is the mobility concept of interest.

One difference from the CIGP, but shared with the TIP curve, is that the height is obtained by dividing by \( n \) rather than \( k \) (where \( k = p \) for the CIGP expression in (3)). However, this apparently small modification is important, since the properties of the TIM curve can more readily illustrate the three mobility characteristics of interest for any population or specified population sub-group. Identifying the inequality of mobility within a given percentile is less straightforward from the CIGP since this requires a visual comparison of (possibly multiple) slope changes across percentiles below \( p \), as in Figure 2.

The TIM curve can be defined more formally as follows, ignoring \( i \) subscripts for convenience. Suppose incomes are described by a continuous distribution where \( H(x_t) \) and \( F(y_t) \) denote respectively the distribution functions of income and log-income at time \( t \), with population size, \( n \). For incomes ranked in ascending order, the TIM curve plots the cumulative proportional income changes, \( y_t - y_{t-1} \), per capita, denoted \( M_{h,t} \), against the corresponding proportion of people, \( h \), where:

\[
h = F(y_{h,t-1})
\]

(4)

Thus \( y_{h,t-1} = F^{-1}(h) \) is the log-income corresponding to the \( h^{th} \) percentile, and the TIM curve plots \( M_{h,t} \), given by:

\[
M_{h,t} = \int_{0}^{y_{h,t-1}} \{ (y_t - y_{t-1}) - (y_{t-1} - y_{t-1}) \} dF(y_{t-1}) + (\mu_t - \mu_{t-1}) F(y_{h,t-1})
\]

(5)

against \( h \).\(^7\)

Let \( \mu_t \) denote the arithmetic mean of log-income (that is, the logarithm of the geometric mean, \( G_t \), of income, \( x_t \)). Then equation (5) can be written as:

\[
M_{h,t} = \int_{0}^{y_{h,t-1}} \{ (y_t - \mu_t) - (y_{t-1} - \mu_{t-1}) \} dF(y_{t-1}) + (\mu_t - \mu_{t-1}) F(y_{h,t-1})
\]

(6)

The term, \( y_t - \mu_t \) is equal to \( \log(x_t/G_t) \). Hence \( (y_t - \mu_t) - (y_{t-1} - \mu_{t-1}) \) is the proportional change in relative income. Thus, \( M_{h,t} \) consists of the cumulative proportional change in income relative to the geometric mean, plus a component that depends only on the proportional change in geometric mean income.

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\(^7\)For very large datasets it is convenient to plot values of the cumulative proportional change corresponding to percentiles, \( P_j \), for \( P_1 = 0.01 \) and \( P_j = P_{j-1} + 0.01 \), for \( j = 2, ..., 100 \). Thus, obtain the cumulative sum \( M_j = \frac{1}{n} \sum_{i=1}^{n} P_j (y_{i,t} - y_{i,t-1}) \), where, as above, \( n \) is the number of individuals in the sample. Hence for \( j = 2, ..., 100 \): \( M_j = M_{j-1} + \frac{1}{n} \sum_{i=n}^{n+P_{j-1}} (y_{i,t} - y_{i,t-1}) \). The TIM curve is then plotted using just 100 values.
Suppose the proportional change in the geometric mean, $\mu_t - \mu_{t-1}$, is equal to $g$. Furthermore, suppose the proportional change in relative income depends on income in $t - 1$, so that $(y_t - \mu_t) - (y_{t-1} - \mu_{t-1})$ can be written as the function, $g^* (y_{t-1})$. Then (6) can be expressed as:

$$M_{h,t} = \int_0^{y_{h,t-1}} g^* (y_{t-1}) dF (y_{t-1}) + gh$$

(7)

If all individuals receive exactly the same relative income change, relative positions are unchanged and $g^* (y_{t-1}) = 0$ for all $y_{t-1}$. Hence, $M_{h,t}$ plotted against $h$ is simply a straight line through the origin with a slope of $g$. This means that the extent of differences in proportional income changes over any range of the income distribution can be seen immediately by the extent to which the TIM curve deviates from a straight line, which in turn depends on the properties of $g^* (y_{t-1})$. Appendix A considers a special case where $(y_t - \mu_t)$ is a linear function of $(y_{t-1} - \mu_{t-1})$, reflecting Galtonian regression towards the (geometric) mean, and a random component.

A hypothetical example of a TIM curve is shown in Figure 4. The particular curve illustrates a situation in which relatively lower-income individuals receive proportional income increases which are greater than that of average (geometric mean) income. Hence the TIM curve, OHG, lies wholly above the straight line OG.

Figure 4: A TIM Curve

If all incomes were to increase by the same proportion, the TIM curve would be the straight line OG. The height, G, indicates the average growth rate of the population as a
whole, while the height, $H$, indicates the average growth rate of those below $x(h)$: these heights reflect intensity properties of mobility. Furthermore, the inequality of growth rates is reflected in the degree of curvature. For example, the curvature of the arc OH relative to the straight line OH indicates that lower-income individuals have higher growth than those individuals to the left of, but closer to, $h$. If concern is for those below a poverty line, $x_P$, the corresponding percentile is $h_P = F(x_P)$, where, as defined above, $F(x)$ is the distribution function of $x$. The TIM curve gives an immediate indication of whether income changes have been pro-poor.

Suppose interest is focussed on those below the $h^{th}$ percentile, indicated in Figure 4. There is less inequality of mobility – a lower dispersion of income changes – among the group below $h$, shown by the fact that the TIM curve from O to H is closer to a straight line than the complete curve OHG.\(^8\) The TIM curve also shows that the income growth of those below $h$ is larger than that of the population as a whole. The average growth rate among the poor – the intensity of their growth – is given by the height $H$.

\[\text{Figure 5: Example of Differential Mobility Pattern}\]

Figure 5 illustrates a TIM curve reflecting a very different pattern of mobility. In this case the lower-income groups experience smaller proportional increases in income than those with higher incomes. If $h_p$ is to the right of the intersection (from below) of the TIM curve with the line OG, average growth of those in poverty exceeds average growth for the population.

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\(^8\)There is a potential ambiguity in the use of the term ‘inequality’ here since the TIP curve refers to a cross-sectional distribution whereas the TIM curve refers to income changes. To avoid confusion the latter could be referred to as the ‘interpersonal dispersion’ of mobility.
as a whole. Yet the convexity of the TIM curve to this point demonstrates that this average pro-poor growth represents quite different experiences among the poor.

4 Positional Mobility

An alternative class of mobility measures is based on the idea of mobility as positional change, rather than relative income change. It is therefore useful to examine whether a diagram similar to the TIM curve can be helpful in this context. This section focuses on income re-ranking positional change. As highlighted by Fields (2000), individuals can move to higher or lower rank positions, so that the explicit treatment of the direction of change becomes important for mobility measurement. Defining a re-ranking mobility index it is therefore first necessary to decide whether negative re-ranking (dropping down the ranking) is treated symmetrically with upward (positive) movement within the ranking. A second issue concerns the choice of whose mobility is to be included.

In the following discussion, individuals are ranked in ascending order of initial incomes, $x_{i,0}$, so that ranks $i = 1, \ldots, n$ are for individuals from the lowest to the highest income. The initial period is denoted 0, and initial ranks may be defined as $R_{i,0} = i$. Consider, as in previous sections, the case where it is desired to measure the extent of mobility of a subset of individuals, $k \leq n$, with the lowest initial incomes, and let $\Delta R_i = R_{i,1} - R_{i,0} = R_{i,1} - i$ denote the change in the rank order of the person who initially has rank, $i$. Three treatments of re-ranking are possible, all related to how negative, or downward, re-ranking is treated. Firstly, negative re-ranking could be treated symmetrically with positive re-ranking such that positional mobility is defined in net terms, that is, positive changes in rank net of any negative changes within group $i = 1, \ldots, k$. This is referred to as ‘net re-ranking’. Secondly, negative movement in the ranking could be ignored, which simply involves setting $\Delta R_i = 0$ when $\Delta R_i < 0$. This is referred to as ‘positive re-ranking’. Thirdly, re-ranking may be measured in absolute terms in which all re-ranking is measured as a positive value. This is referred to as ‘absolute re-ranking’.

As Fields and Ok (1996) and Fields (2000) stress, the appropriate choice among these three measures depends on the question of interest. For example, if interest is focussed on those below the poverty line as a group, then it may be desirable to balance any upward mobility by some of those in poverty with downward mobility of others in poverty, in order to gain an indication of the overall experience of the group. This suggests a focus on net

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9If individual changes in rank are simply aggregated to obtain an aggregate mobility index, then a change in rank of 50 places by one individuals is treated symmetrically with 50 individuals each changing one ranking place.
mobility in this case. Likewise, if movement per se is the mobility concept of interest, then a non-directional measure such as absolute re-ranking is more relevant. Positive re-ranking quantifies only those who are moving up, a common metric when assessing the persistence of low income or poverty status for a sub-set of individuals or households.

The three re-ranking indices for an individual initially having rank order, \( i \), (for \( i = 1, \ldots, n \)) are defined formally as follows:

\[
M_i^{net} = \Delta R_i
\]

\[
M_i^{pos} = \Delta R_i \mid_{\Delta R_i > 0}
\]

\[
M_i^{abs} = |\Delta R_i|
\]

Cumulated across the \( k \) lowest income individuals in period 0, the corresponding aggregate re-ranking indices are given by:

\[
M_k^{net} = \sum_{i=1}^{k} M_i^{net} = \sum_{i=1}^{k} (R_{i,1} - R_{i,0})
\]

\[
M_k^{pos} = \sum_{i=1}^{k} M_i^{pos} = \sum_{i=1}^{k} (R_{i,1} - R_{i,0}) \quad \text{for} \quad \Delta R_i > 0
\]

and

\[
M_k^{abs} = \sum_{i=1}^{k} M_i^{abs} = \sum_{i=1}^{k} |R_{i,1} - R_{i,0}|
\]

The absolute re-ranking case may be thought of as describing the extent of overall positional change within the relevant range of the income distribution.

To examine the ‘three Is’ properties similar to the TIM, but based on the indices in (11), (12) and (13), one approach would be to plot the value of the relevant \( M_k \) index against the cumulative fraction of the population, \( h = k/n \). However, there are two difficulties with the indices in (11) to (13). Firstly, they are not scale independent, since they depend on \( k \) and hence population size: more re-ranking is possible in larger populations. One solution would be to scale the three \( M_k \) indices by \( n \). However, as shown below, a slightly different rescaling, by \( (n/2)^2 \), yields ‘normalised’ values, \( m_k \), that lie between zero and one (or zero and two for absolute re-ranking). These may then readily be plotted against \( 0 \leq h \leq 1 \).

Secondly, an individual’s opportunity for re-ranking is partly determined by the initial position in the income ranking: those among the lowest ranks have less opportunity to move down, other things equal, than those higher up, and vice versa. It is therefore useful to consider the maximum re-ranking possible for each individual; actual re-ranking may then be compared with these maximum values for any given \( h \).
On maximum re-ranking, to simplify the exposition consider a population of \( n = 100 \) individuals, each with a different income level. Hence each integer, \( i = 1, \ldots, n \), represents a percentile of the distribution. In period 0, they are ranked, \( R_{i,0} = 1 \ldots 100 \), representing the lowest to the highest incomes. Two polar cases are the maximum and minimum degrees of mobility possible. The former is defined here as a complete ranking reversal, \( \Delta R_{i}(\text{max}) \), such that in period 1, \( R_{i,1} \) involves a lowest to highest ranking, and \( R_{i,1} \ (\text{max}) = n + 1 - R_{i,0} = 100, \ldots, 1 \).

Maximum re-ranking implies:

\[
M_{i}(\text{max}) = \Delta R_{i}(\text{max}) = R_{i,1}(\text{max}) - R_{i,0} = n + 1 - 2R_{i,0} \quad (14)
\]

For large \( n \), this is approximated by \( n - 2R_{i,0} \). Where it is desired to measure the extent of re-ranking of the subset of individuals, \( k \leq n \), with the lowest incomes, the cumulative maximum re-ranking index for the net mobility case, \( M_{k}^{\text{net}}(\text{max}) \), is given by:

\[
M_{k}^{\text{net}}(\text{max}) = \sum_{i=1}^{k} M_{i}^{\text{net}}(\text{max}) = \sum_{i=1}^{k} (n + 1 - 2R_{i,0}) \quad (15)
\]

Using the sum of an arithmetic progression, whereby \( \sum_{i=1}^{k} R_{i,0} = 1 + 2 + \ldots + k = k(k+1)/2 \), equation (15) becomes:

\[
M_{k}^{\text{net}}(\text{max}) = \sum_{i=1}^{k} (n + 1 - 2R_{i,0}) = k(n + 1) - k(k + 1)
= k(n - k) \quad (16)
\]

Hence, in the above example with \( n = 100 \), if interest focuses only on the poorest individual \((k = 1)\), maximum net re-ranking is given by \( M_{k}^{\text{net}}(\text{max}) = (100 - 1) = 99 \); when \( k = 2 \), \( M_{k}^{\text{net}}(\text{max}) = 2(100 - 2) = 196 \); and so on. More generally, since maximum re-ranking (complete ranking reversal) involves all those below the median individual changing positions with those above the median, it follows from (16) that the maximum value of \( M_{k}^{\text{net}}(\text{max}) \) as \( k \) increases is obtained for \( k = n/2 \), yielding \( M_{k}^{\text{net}}(\text{max}) = (n/2)^{2} \).

\(^{10}\)An alternative argument proposes that the relevant comparator should be defined as when the change in an individual’s position in the ranking is purely random; see Jantti and Jenkins (2015, pp. 8-9). That is, ‘maximum’ mobility involves independence from initial positions, rather than complete reversals. In that case, given \( R_{i,0} \), maximum mobility requires an actual ordering in period 0 to be compared with a random ordering in period 1. Jantti and Jenkins reject the use of ‘maximum’ when mobility is based on origin independence because, they suggest, “it is difficult to argue that origin independence represents ‘maximum’ mobility in the literal sense”. Trivially, the minimum degree of re-ranking involves no change in the ranks such that \( R_{i,1}(\text{min}) = R_{i,0} \) for all \( i \), hence \( \Delta R_{i} = 0 \).

\(^{11}\)Strictly, for small \( n \), the median individual is \( k = (n+1)/2 \), and \( M_{k}^{\text{net}}(\text{max}) \) is given by \((n+1)(n-1)/4\).
This measure serves to highlight the scale dependence of both \( M^\text{net} \) and \( M^\text{net}(\text{max}) \): larger populations imply larger values of both indices. These could be normalised to create a form of per capita index by dividing by \( n^2 \) such that, from (16), the index becomes:

\[
m^\text{net}(\text{max}) = h(1-h).
\]

The maximum value would be reached at \( h = 0.5 \), where \( m^\text{net}(\text{max}) = 0.25 \). However, to yield a more readily illustrated index with a maximum value of one, at \( k = n/2 \), it is preferable to divide by \( (n/2)^2 \). That is,

\[
m^\text{net}(\text{max}) = 4M^\text{net}(\text{max})/n^2 \tag{17}
\]

and, using (16):

\[
m^\text{net}(\text{max}) = 4h(1-h) \tag{18}
\]

A similar exercise for positive re-ranking, \( M^\text{pos}(\text{max}) \), shows that the value of \( M^\text{pos}(\text{max}) \) also reaches a maximum as \( k \) increases of \( M^\text{pos}(\text{max}) = n^2/4 \) when \( k = n/2 \), since all individuals below \( n/2 \) experience positive re-ranking in this (maximum) case. However, above \( k = n/2 \), as more above-median individuals are included within \( k \), their re-rankings are now given by \( \Delta R_i = 0 \), such that the cumulative index, \( M^\text{pos}(\text{max}) \), remains unchanged as \( k \to n \). Thus a similarly rescaled \( m^\text{pos}(\text{max}) \) may be defined analogously to (17) to yield a positive re-ranking index where \( 0 \leq m^\text{pos}(\text{max}) \leq 1 \).

Finally, for the absolute re-ranking case in (13), \( M^\text{abs}(\text{max}) \), it can be shown that, as with the other cases, this increases as \( k \) increases from \( k = 1 \) to \( k = n/2 \) to reach \( M^\text{abs}(\text{max}) = (n/2)^2 \). However, this represents a point of inflection rather than a maximum, since inclusion of the absolute value of above-median individuals’ re-ranking in \( M^\text{abs}(\text{max}) \), ensures that \( M^\text{abs}(\text{max}) \) continues to increase for \( k > n/2 \), reaching \( M^\text{abs}(\text{max}) = n^2/2 \) at \( k = n \). As a result an absolute re-ranking index \( m^\text{abs}(\text{max}) \) obtained by rescaling by \( (n/2)^2 \) lies between zero and two.

Finally, to compare actual and maximum re-ranking mobility, the expressions for actual mobility in (11) to (13) can be similarly rescaled or normalised by \( (n/2)^2 \) to obtain actual aggregate re-ranking mobility expressions, \( m^\text{net} \), \( m^\text{pos} \), and \( m^\text{abs} \), given in each case by:

\[
m = 4M_k/n^2 \tag{19}
\]

Thus, \( 0 \leq m^\text{net}, m^\text{pos} \leq 1 \) and \( 0 \leq m^\text{abs} \leq 2 \). This suggests a convenient illustrative device for positional mobility, a cumulative re-ranking curve, similar to the TIM curve for relative income mobility, that plots \( m_k \) against \( h \). This is explored in the next section using income data for three large longitudinal samples of New Zealand individual taxpayers over 1998 to 2010. First, the next subsection shows \( m_k(\text{max}) \) profiles and introduces an alternative illustration based on the ratio of actual to maximum re-ranking: the re-ranking ratio, \( RRR \).
4.1 Maximum Re-Ranking profiles

Profiles for the three (rescaled) maximum re-ranking cases discussed above, \( m^\text{net}_k(\text{max}) \), \( m^\text{pos}_k(\text{max}) \), and \( m^\text{abs}_k(\text{max}) \), are plotted against \( h = k/n \) in Figure 6. This shows the distinct non-linear shape of the maximum profiles, whichever definition of positional mobility is adopted. As expected, the net re-ranking profile displays a parabolic shape which differentiation of (18) reveals has a slope of \( 4(1 - 2h) \), that equals zero at \( h = 0.5 \) (the 50\(^{th}\) percentile), thereafter declining symmetrically to a slope of \(-4\) at \( h = 1 \). The equivalent positive re-ranking profile also reaches a maximum at the 50\(^{th}\) percentile but remains constant thereafter, while the absolute re-ranking profile displays a sigmoid shape, reaching a local point of inflection where \( m^\text{abs}_k(\text{max}) = 1 \) at the 50\(^{th}\) percentile, but then rising at an increasing rate till \( m^\text{abs}_k(\text{max}) = 2 \) at \( h = 1 \).

The maximum re-ranking indices in Figure 6 are invariant to population size, but they vary with the population percentile, of interest, \( h \). Thus, the scope or opportunity for a given degree of actual re-ranking clearly also varies with \( h \). A natural index of interest therefore is the ratio of actual to maximum mobility at each percentile, \( h \). This is referred to as the re-ranking ratio, \( RRR_k \), and can be calculated for net, positive and absolute re-ranking.
For example, the net re-ranking case is given by:

\[
RRR_{k}^{\text{net}} = \frac{m_{k}^{\text{net}}}{m_{k}^{\text{net}}(\text{max})} = \frac{M_{k}^{\text{net}}}{M_{k}^{\text{net}}(\text{max})}
\]  

(20)

where the numerator and denominator are given respectively by (19) and (17), or by (11) and (16). This ratio can also be plotted against \( h \) to identify how the extent of mobility changes by cumulative percentile of the population relative to the maximum possible for that percentile.

Recognising these differences in maximum re-ranking is important when interpreting differences in actual re-ranking for different values of \( h \). In particular, a smaller value of \( m_{k}^{\text{net}} \) at \( h = 0.1 \), compared to \( m_{k}^{\text{net}} \) at \( h = 0.3 \), for example, may be partly or entirely due to the fact that individuals up to \( h = 0.1 \) cannot achieve the higher \( m_{k}^{\text{net}} \) observed at \( h = 0.3 \), even in the absence of other constraints on re-ranking mobility.

Section 6 illustrates these re-ranking mobility measures using data on New Zealand taxpayers. First, section 5 uses the same data to construct TIM curves.

5 TIM Curves for New Zealand

This section illustrates the TIM curves described in section 3, using data from a 2 per cent random sample of individual New Zealand Inland Revenue personal income taxpayers. Using data for 1998, 2002, 2006 and 2010, three separate panels were obtained for 1998 to 2002, 2002 to 2006 and 2006 to 2010, each five-year panel containing incomes for both years for the same taxpayers.

To avoid the exercise being contaminated by taxpayers with very low incomes, such as small part-time earnings of children, or small capital incomes of non-earners, individuals with annual incomes less than \$1,000 were omitted from the sample. This yielded usable samples of \( n = 29,405 \), 31,355 and 32,970 individuals respectively for the three five-year panels. In each case individuals were ranked by their initial year incomes, with all of the diagrams below showing cumulative percentiles of the income distribution, \( h \), in the relevant initial year (1998, 2002, or 2006) on the horizontal axis. Appendix B provides further details on the New Zealand income mobility data.

Figure 7 shows the three TIM curves. Growth rates shown on the vertical axis are measured over the five-year period. The right-hand end of the TIM curve represents the average growth rate (over the five years) across all \( n \) individuals. This was similar, at around 15 per cent, for the periods 1998 to 2002 and 2006 to 2010, but was around 20 per cent, over the period 2002 to 2006. All three curves rise steeply at the lowest income percentiles and
flatten out at higher percentiles, yielding relatively concave profiles and suggesting greater upward mobility especially among the lower percentiles. These curves combine average growth (across all individuals) and relative growth, but the latter can most easily be seen by normalising each curve using the sample average growth rate.

Figure 8 shows the normalised TIM curves, which end at a normalised cumulative growth rate of one. These allow the curvature of each profile to be more easily compared. The 2002 to 2006 normalised TIM reveals unambiguously lower relative mobility than the other two curves, for all \( h \). There appears to be a clear ranking according to the extent of relative mobility, with 1998 to 2002 exceeding that of 2006 to 2010 and the latter exceeding 2002 to 2006 which displays less curvature, implying more equality in income growth, at any selected percentile.\(^\text{12}\)

Figure 8 has been constructed to illustrate the extent of interpersonal inequality of mobility for the sample as a whole, by comparing the concavity of the three normalised TIM curves to the straight line representing equality of mobility. However, if interest is focused on the inequality of mobility for a particular income group, such as the poorest half of the sample, so that \( h = 0.5 \), then each TIM curve can be re-normalised using the average income growth rate for this group.

Identifying the underlying influences that might explain the different mobility patterns

\(^\text{12}\)Section 5.1 examines how far these conclusions hold statistically when allowing for sampling variability.
Figure 8: Normalised TIM Curves for New Zealand

observed across the three periods in Figure 8 is beyond the scope of the present paper. However it is noteworthy that the period revealing least relative mobility, 2002-06, is also the period with higher average income growth, as seen in Figure 7, and without major macroeconomic shocks. The other two periods experienced significant shocks, such as the short New Zealand downturn in 1998-99 and the major global financial crisis in 2008-10.13

Such aggregate shocks may be expected to be associated with greater income volatility at the individual (or firm) level, and hence contribute to greater measured individual income mobility. Given the prominence of the agricultural sector in the New Zealand economy, the country is also especially sensitive to weather-related impacts on incomes in agriculture and downstream industries. As New Zealand Treasury (2015, p. 20) shows, drought conditions were experienced during 1998, 2008 and 2010, but not in the intervening years. These patterns may have contributed to the greater relative mobility of incomes over 1998-2002 and 2006-2010 but less so in 2002-2006.

5.1 Sampling Variability

When comparing Lorenz, concentration or TIP curves for different periods or population groups, it is not usual to examine the statistical properties of these simple graphical devices.

13For more details see, for example, Carroll (2012). Reserve Bank of New Zealand (2008; Figure 4) also shows that aggregate output volatility fell substantially in New Zealand during the mid-2000s.
Indeed, their attractiveness lies in the fact that they can reveal important features ‘at a glance’ and are very easy to produce. Nevertheless, for present purposes it is useful to know the extent to which such diagrammatic comparisons, based on sample data, are likely to allow robust conclusions regarding mobility characteristics. This section therefore considers the sample variation around each of the TIM curves reported above to assess how far the position and shape of each TIM curve may be subject to sampling error. With estimates for each TIM in Figure 8 based on around 30,000 individuals, drawn randomly from approximately three million New Zealand income taxpayers, sampling errors can be expected to be small.

For each normalised TIM curve, confidence intervals around the sample TIM curve can be computed using bootstrap methods.\footnote{Jenkins and Van Kerm (2016) mainly focus on confidence intervals around estimated differences between any two periods IGP\textsuperscript{s} or CIGP\textsuperscript{s}; see their Figure 2. This reflects their primary concern over the robustness of welfare dominance conclusions. For the TIM curves presented here, interest is as much related to the robustness of the \textit{shape} of the curve (reflecting inequality of mobility across the income distribution) as to differences from TIM\textsuperscript{s} for other periods. Bootstrapping is therefore applied at various percentiles for each TIM rather than for inter-period TIM differences, though clearly the latter could also readily be tested.} This generates standard errors and error bands associated with, for example, the 5\textsuperscript{th} and 95\textsuperscript{th} percentile of the distribution of bootstrapped estimates.\footnote{The bootstrapped 5 and 95 percentile band, rather than taking plus or minus two standard errors around the TIM, are preferred here because the distribution of the estimates appears not to be Normal. That is, the 5 and 95 percentiles band is narrower, and slightly asymmetric compared to that obtained using two standard errors, due to a more concentrated, slightly skewed distribution. Also, unlike Jenkins and Van Kerm (2016) who were concerned about sample dependence between their different period IGP\textsuperscript{s}, the independent random sampling of around 1 per cent of New Zealand income taxpayers for each of the sets of samples used here, the probabilities of individuals being repeatedly selected in 1998, 2002 and 2006 are low.} The bootstrap method proceeded as follows. Using the initial income-ranked distribution of \( n \) individuals, \( n \) random draws were obtained to yield one complete bootstrap replicate of the distribution of individuals with their associated income growth rates.

The properties (initial income levels, growth rates, TIM value) of the individual at each of twenty ventiles (0.05, 0.10, 0.15 percentiles, and so on) were obtained for this replicated sample. This process is repeated 100 times with the standard deviation of the generated sampling distribution (the estimated standard errors) and 5\textsuperscript{th} and 95\textsuperscript{th} percentiles at the twenty ventiles being recorded after each additional ten replications. This revealed how far the estimates are sensitive to extending the number of replications. It was found that the standard errors and relevant percentiles stabilised well before 100 replications.

Figure 9 shows the outcome of this process for the 2002-06 and 2006-10 normalised TIM\textsuperscript{s} with their associated 5\textsuperscript{th} and 95\textsuperscript{th} percentile error bands obtained at each ventile.\footnote{Bands based on plus or minus two standard deviations are slightly larger and, of course, symmetrical}
Figure 9: TIM Curves and Sampling Variability

Two features are immediately clear from Figure 9. Firstly, confidence intervals for the two TIMs shown are small, such that they do not overlap for any ventile up to around the 80th percentile. Secondly, the error bands generally widen at higher ventiles. This aspect, which arises from the cumulative nature of the TIM with increasing ventiles, is less prominent when confidence intervals are considered as ratios of the TIM values at each ventile. Nevertheless, some widening remains evident, as \( h \) increases.\(^{17}\)

5.2 Comparisons with Cumulative Income Growth Profiles

As mentioned in Section 2.3, the TIM curve is closely related to the CIGP proposed by Jenkins and Van Kerm (2016) and Bourguignon’s (2011) na-GIC curve: the TIM is obtained by dividing cumulative growth rates at each percentile, \( p \), by the sample size, \( n \), rather than by \( p \). It was suggested that this conveniently allowed the inequality of mobility among those below \( p \) to be illustrated alongside incidence and intensity properties.

\(^{17}\)For example, based on the 2006-10 normalised TIM, at \( h = 0.05 \), the standard error is 1.7 per cent of the TIM value. This generally rises with \( h \), reaching 2.5 per cent at \( h = 0.95 \). This result is also observed for 2002-06.
Differences between the TIM curve in Figure 7 and a CIGP can be seen in Figure 10, using data for the 2006 to 2010 panel of taxpayers. In this case, because of the large sample size for New Zealand, compared with Jenkins and Van Kerm’s UK data (reproduced in Figure 2 above), the CIGP is much smoother, and nicely illustrates the (mostly) declining cumulative growth rates as higher initial income individuals are added to the profile, that is, as $p$ increases. However, the CIGP also illustrates that if some of the lowest-income individuals have zero or low income growth, as in this case, the CIGP starts from the origin and rises rapidly before declining as $p$ increases.

This makes judgements regarding the inequality of mobility among those below any chosen $p$ more difficult, especially at low percentiles, captured in this case by the shape of the profile, relative to the $x$-axis, from the origin to the $p^{th}$ percentile of interest. Inequality of mobility is more readily assessed using the TIM curve, from the concavity of the curve relative to a straight line from the origin to the point on the curve at the $p^{th}$ percentile of interest. Nevertheless, both curves provide useful illustrations of the diversity of growth across the initial income distribution and the strong overall ‘regression to the mean’ characteristic.
6 Re-ranking Profiles for New Zealand

This section turns to an application of the positional mobility measures described in section 4 to the same New Zealand income data, to assess both the extent of observed positional mobility and the incidence, intensity and inequality dimensions. This is illustrated first by plotting the re-ranking measures $m_k^{\text{pos}}$, $m_k^{\text{net}}$ and $m_k^{\text{abs}}$ against $h$, analogous to the $m_k(\text{max})$ profiles in Figure 6. These are shown for the 2006 to 2010 period in Figure 11. In each case, these profiles could contain concave, linear or convex segments, reflecting the degree of re-ranking being experienced as $h$ is increased to include higher income individuals. A greater amount of re-ranking mobility generates profiles that are more concave or less convex.

To assess the incidence, intensity and inequality aspects of these re-ranking measures, Figure 11 can be interpreted as follows. For a given definition of positional mobility (net, positive or absolute re-ranking), select a value of $h$ representing the sub-set of low income individuals of interest (the incidence dimension). The height of the profile on the vertical axis at this value of $h$ represents the intensity of re-ranking for this group, namely how much re-ranking they have experienced on average. The section of the profile to the right of $h$ becomes irrelevant, equivalent to the flat section of the TIP curve.
Figure 12: Actual Positive Re-ranking: Three Periods

The deviation from linearity (concave or convex) of the $m_k$ profile, from the origin to its value at the selected $h$, provides a measure of the extent of inequality of mobility within $h$. That is, the actual profile may be compared to a straight line from the origin to the relevant value of $m_k$. For example, in Figure 11 the profile for absolute re-ranking appears to be approximately linear over a wide range above the $10^{th}$ percentile. This suggests that, at least for this sample and measure, the extent of re-ranking is relatively constant across the income distribution.

As with the TIM curves in Section 5, changes in the incidence, intensity and inequality of positional mobility associated with different time periods can be examined by plotting relevant $m_k$ profiles for the three periods. Figure 12 illustrates this for the positive re-ranking measure, $m_k^{pos}$, showing that the characteristics of positive re-ranking mobility across the three periods are very similar, both in terms of levels of $m_k$ at each value of $h$, and the degree of inequality of mobility (concavity) of each profile for any given $h$.

To the extent that the profiles differ, there is some evidence of slightly more re-ranking mobility during the first period, 1998-2002, as observed for the income growth based TIM curves.\textsuperscript{18}

\textsuperscript{18}Since the maximum positive re-ranking for $h \geq 0.5$, is equal to one (see Figure 6), the values of $m_k$ in
While some groups across the sample may experience higher re-ranking in Figure 11, their movements are constrained to differing degrees by the maximum re-ranking possible. The differences between the actual $m_k$, and the equivalent $m_k(\text{max})$, can be identified by considering changes in $RRR_k$ as $h \rightarrow k$. Relevant profiles plotting $RRR_k$ against $h$ are shown for the three re-ranking measures in Figure 13, where values on the vertical axis are simply ratios of the axis values in Figures 11 and 6.

This indicates that, for all three re-ranking measures in the New Zealand case, the extent of positional mobility relative to the maximum achievable is relatively high for the lowest-income individuals (low $h$), at around 0.25 – 0.3. This steadily declines as $h$ is increased, to a minimum of approximately 0.2 at around the 20$^{th}$ to 25$^{th}$ percentile. Thereafter, the $RRR_k^{\text{abs}}$ rises to around the 70$^{th}$ percentile, while the $RRR_k^{\text{pos}}$ profile rises to the 100$^{th}$ percentile. From this it may be inferred that the group experiencing absolute re-ranking that is closest to the maximum achievable is the group between approximately the 50$^{th}$ and 70$^{th}$ percentiles. For positive re-ranking, actual and maximum re-ranking are generally closest for the lowest and highest population percentiles, reaching around $RRR_k^{\text{pos}} = 0.3$. The exception is the case of the $m^\text{net}_k$ profile which continues to decline for $h > 0.2$, though

Figure 12 also reveals the values of the re-ranking ratio, $RRR_k = m_k/m_k(\text{max})$ for $h \geq 0.5$. 

25
at a somewhat slower rate than for $h < 0.2$.\footnote{The strong fluctuations in the $M_{h}^{net}$ curve as $h$ approaches 1, reflect the fact that the value of both the actual and maximum net re-ranking measures equal zero at $h = 1$. Hence the ratio can be quite unstable in the vicinity of $h = 1$. (and is, of course, undefined at $h = 1$.)}

The fact that $RRR_{k}^{pos}$ and the $RRR_{k}^{abs}$ profiles reach the same value for $h = 1$ is not coincidental. It has already been shown that $M_{k}^{bs}(max) = n^2/2$, while $M_{h}^{pos}(max) = n^2/4$, at $h = 1$; that is, $M_{h=1}^{abs}(max) = 2M_{h=1}^{pos}(max)$ and hence $m_{h=1}^{abs}(max) = 2m_{h=1}^{pos}(max)$. This same relationship holds for the actual measures: $m_{h=1}^{abs} = 2m_{h=1}^{pos}$. This can be seen by noting that:

$$M_{h=1}^{abs} = \sum_{1}^{n} |R_{i,1} - R_{i,0}|$$

However, at $h = 1$ the sum of positive ranking movements must equal the sum of negative ranking movements, so that:

$$M_{h=1}^{abs} = 2\sum_{1}^{n} (R_{i,1} - R_{i,0}) \mid_{\Delta R_{i} > 0}$$

The term after the summation in (22) is simply the positive re-ranking measure, $M_{h=1}^{pos}$. Hence the $RRR_{k}$ for both the positive and absolute re-ranking measures are equal at $h = 1$.

Considering the three profiles in Figure 13, the measure of net movement, $RRR_{k}^{net}$, indicates a persistent downward trend as $h$ approaches $1$. This suggests that low-income individuals generally experienced more movement in their income rank (relative to the maximum achievable) over this period compared with those on higher incomes. This seems likely to be capturing a re-ranking analogue of the ‘regression to the mean’ in income levels observed above.

Figure 14 shows profiles for $RRR_{k}^{pos}$ for the three periods, equivalent to the three $m_{k}^{pos}$ profiles in Figure 12. This reveals considerable volatility in $RRR_{k}$ over the lowest 5 percentiles, perhaps not surprisingly given the numbers of individuals with low incomes in the initial year and who experience a wide range of income changes over the period.\footnote{For example, in 2006, the 5th percentile income level for the 2006-10 panel is only around $6,600; see Appendix B.} Much of this mobility probably reflects some low-income individuals, such as secondary earners, moving into employment or from part-time to full-time work, while others remain in their initial employment status. These data also include the self-employed who are known to experience greater income volatility.

Above the 5th percentile, the profiles behave similarly to the $m_{k}^{pos}$ profiles in Figure 12, with generally greater re-ranking as a fraction of the maximum possible in 1998-2002. However, for percentiles around the 5th to the 40th, the 2006 to 2010 $RRR_{k}^{pos}$ profile is more
Figure 14: Positive Re-ranking Ratios: Three Periods

clearly below the 2002 to 2006 equivalent. This was less obvious for the $m_k^{pos}$ profiles in Figure 12. These results therefore generally confirm that, across most percentiles of the initial income distribution, re-ranking mobility was slightly greater in the earliest period examined, 1998-2002, and decreased somewhat in the two subsequent periods. They also demonstrate that, across all three periods, re-ranking is typically around 20-30 percent of the maximum mobility possible, conditional on an individual’s position in the initial income distribution. It also tends to be highest at both the top and at the bottom of those initial distributions.

7 Conclusions

Over two decades ago, Jenkins and Lambert (1997) introduced new insights into the poverty measurement literature by demonstrating that the incidence, intensity and inequality of poverty could be illustrated by their ‘Three Is of Poverty’ (TIP) curve. These aspects are also captured, in specific ways, by various summary measures of poverty. This paper has suggested that, for mobility concepts based on income growth and re-ranking, the same three important dimensions can be translated to the context of mobility – incidence, intensity and
inequality – and are readily and simultaneously identifiable using new illustrative devices proposed here. The advantage is that, like the Lorenz curve in the case of static inequality, they are simple to produce and provide convenient comparisons of the different dimensions, and can be suggestive of further analysis.

Specific summary measures (of inequality, poverty, mobility and so on) are inevitably needed to supplement the diagrams. But the measures necessarily involve a loss of information: for example, a wide range of Lorenz curves are consistent with the same Gini inequality measure. The simplicity, the immediate visibility of essential characteristics, the ability to deal with the detail revealed over the whole range of incomes, and the need to communicate such important concepts to a non-technical audience, explain the popularity of diagrammatic methods.

For income mobility measured as relative income growth, based on an analogue of the TIP curve, this paper has proposed that a ‘Three Is of Mobility’, or TIM, curve can provide a useful means of combining and illustrating these three concepts within a single diagram. This plots the cumulative proportion of the population (from lowest to highest values of initial income) against the cumulative change in log-incomes per capita over a given period.

For mobility measures based on positional changes, or the extent of re-ranking of individuals over a given period, it was shown that an equivalent re-ranking mobility curve can illustrate the incidence, intensity and inequality of positional mobility in the form of re-ranking. This plots the cumulative degree of re-ranking against the cumulative proportion of the population (from lowest to highest incomes). Additionally, since for any given fraction of the population there is a different maximum possible extent of re-ranking, it is useful to consider the cumulative re-ranking ratio of actual-to-maximum re-ranking against the cumulative proportion of the population.

Illustrations for both of these mobility concepts – relative income growth and re-ranking – were examined based on three panels of New Zealand incomes from 1998 to 2010. These showed that income growth rates within the lower part of the income distribution were quite substantially higher than those observed higher up the income distribution, reflecting in part a relatively high degree of regression towards the mean.\(^{21}\) These TIM curves provide a convenient counterpoint to evidence from cross-sectional distributions over various periods by Perry (2017; chapter D) based on household data from the late 1980s. That suggested similar growth rates across income deciles, or lower (higher) growth for the lowest (highest) deciles - a quite different cross-sectional result from the longitudinal evidence here.

\(^{21}\)Van Kerm (2009) and Jenkins and Van Kerm (2011, 2016) report similar regression to the mean patterns in their income growth profiles for the UK and a selection of other European countries.
Evidence on the extent of re-ranking of individual incomes across a five year period also suggested a relatively high degree of positional mobility, compared to the maximum possible, among the lowest and highest income individuals. This highlighted how some conclusions regarding the extent of re-ranking depends crucially on the re-ranking measure adopted – positive, net or absolute. For example, the highest re-ranking ratios are observed around the 50th to the 70th percentiles for an absolute re-ranking measure but rise steadily towards the 100th percentile when a positive re-ranking ratio is considered.

While users of TIM curves, just like users of Lorenz curves, will typically not wish to undertake the cumbersome task of constructing confidence intervals, the present paper has shown that 95 per cent confidence intervals around the TIM curve, for the kind of samples used here, are narrow. Hence qualitative comparisons of mobility using such curves, for different periods or population groups, can be made with reasonable confidence.
Appendix A: The TIM Curve and Galtonian Regression to the Mean

It was shown in section 5 that the TIM curve can be written as:

\[ M_{h,t} = \int_0^{y_{h,t-1}} g^* (y_{t-1}) dF (y_{t-1}) + gh \]  \hspace{1cm} (A.1)

where \( g^* (y_{t-1}) = (y_t - \mu_t) - (y_{t-1} - \mu_{t-1}) \) represents the proportional change in relative income. A simple special case is to suppose that:

\[ g^* (y_{t-1}) = -\gamma (y_{t-1} - \mu_{t-1}) + u_t \]  \hspace{1cm} (A.2)

where \( u_t \) is a stochastic term with expected value of zero. For \( \gamma > 0 \), those with \( y_{t-1} > \mu_{t-1} \) experience a systematic relative reduction in income plus a random proportional change. Conversely, those below the geometric mean experience systematic relative income increases. Hence, letting \( 1 - \gamma = \beta \), it can be seen that:

\[ y_t - \mu_t = \beta (y_{t-1} - \mu_{t-1}) + u_t \]  \hspace{1cm} (A.3)

This is the standard Galtonian ‘regression to the mean’ specification; see Creedy (1985). The extent to which \( \beta \) is less than 1 indicates the degree to which those below the geometric mean experience, on average, a higher relative income increase than those above the geometric mean. If, instead, \( \gamma < 0 \), clearly \( \beta > 1 \) and there is regression away from the geometric mean.

Substituting for \( y_t - \mu_t \) from (A.3) into (6) gives:

\[
M_{h,t} = \left[ (\beta - 1) \int_0^{y_{h,t-1}} (y_{t-1} - \mu_{t-1}) dF (y_{t-1}) \right] \\
+ \left[ (\mu_t - \mu_{t-1}) F (y_{h,t-1}) \right] \\
+ \left[ \int_0^{y_{h,t-1}} u_t dF (y_{t-1}) \right]
\]  \hspace{1cm} (A.4)

The height of the TIM curve, at any value of \( h \), referred to here as the ‘intensity’ of mobility associated with the ‘incidence’, \( h \), is thus made up of three components, each contained within square brackets. The first term is \((\beta - 1)\) multiplied by the sum, up to \( y_{h,t-1} = F^{-1} (h) \), of the differences between log-income and mean log-income in period \( t - 1 \) (or the sum of the logarithms of relative income, \( x_i/G_t \)). The second term is \( h \) multiplied by the overall growth rate of (geometric mean) income: this term has a linear profile. The third term is the sum of the stochastic terms. For values of \( y_{t-1} < \mu_{t-1} \) (incomes below the
geometric mean), the slope of \( M_{h,t} \) is positive. A turning point occurs for \( y_{h,t-1} = \mu_{t-1} \), after which the slope is negative, since \( y_{t-1} > \mu_{t-1} \) and \( \beta < 1 \).

Consider the component of the TIM curve, \( M_{h,t}^R \), say, that reflects only the systematic component of relative income changes, the regression towards the mean. Then:

\[
M_{h,t}^R = (\beta - 1) \int_0^{y_{h,t-1}} (y_{t-1} - \mu_{t-1}) \, dF(y_{t-1})
\]  

(A.5)

Furthermore, let \( F_1(y) \) denote the first moment distribution function of log-income, the proportion of total log-income obtained by those with log-income below \( y \). Hence a graph of \( F_1(y) \) plotted against \( F(y) \) gives the Lorenz curve of log-income, with \( F_1(y) \leq F(y) \). Then:

\[
M_{h,t}^R = (1 - \beta) \mu_{t-1} \{ F(y_{h,t-1}) - F_1(y_{h,t-1}) \}
\]  

(A.6)

Given that \( F(0) = F_1(0) \) and \( F(\infty) = F_1(\infty) \), this component of the TIM curve starts and ends at zero. Differentiating:

\[
\frac{dM_{h,t}^R}{dF(y_{h,t-1})} = (1 - \beta) \mu_{t-1} \left( 1 - \frac{dF_1(y_{h,t-1})}{dF(y_{h,t-1})} \right)
\]  

(A.7)

The slope of \( M_{h,t}^R \) therefore depends on the degree of regression, \( 1 - \beta \), and the slope of the Lorenz curve of income in \( t - 1 \) at the corresponding value of \( h = F(y_{h,t}) \). Up to the arithmetic mean of log-income, the slope of the Lorenz curve, \( dF_1/dF \), is less than 1, and above the mean the slope is greater than 1. The curvature, reflecting the ‘inequality’ of income growth rates, is given by:

\[
\frac{d^2M_{h,t}^R}{dF(y_{h,t-1})^2} = -(1 - \beta) \mu_{t-1} \frac{d^2F_1(y_{h,t-1})}{dF(y_{h,t-1})^2}
\]  

(A.8)

More regression towards the mean, resulting from a lower value of \( \beta \), means that the profile is concave and deviates further from a straight line. It also lies everywhere above the profile obtained from a higher \( \beta \) (reflecting less regression to the geometric mean). The maximum height of this component of the TIM curve is obtained by setting (A.7) equal to zero, and recognising the well-known property of a Lorenz curve that its slope, \( \frac{dF_1(y_{h,t-1})}{dF(y_{h,t-1})} \), equals 1 at the point on the curve corresponding to the mean, \( \mu_{t-1} \).

\[
(1 - \beta) \mu_{t-1} \{ F(\mu_{t-1}) - F_1(\mu_{t-1}) \}
\]  

(A.9)

The term in curly bracket is clearly positive, given that the Lorenz curve lies below the diagonal of equality, and hence low \( \beta \) is associated with a higher maximum height of the

\[\text{footnote text}\]

The tangent to the Lorenz curve corresponding to \( \mu_{t-1} \) is parallel to the 45 degree line of equality.
TIM curve. The term in curly brackets is the maximum vertical distance between the Lorenz curve of log-income and the diagonal of equality.

The slope of a ray from the origin to a point on the $M^R_n$ component of the TIM curve is:

$$ (1 - \beta) \mu_{t-1} \left( 1 - \frac{F_1(y_{b,t-1})}{F(y_{b,t-1})} \right) $$

(A.10)

and this of course is always positive. This slope depends on the extent of regression towards the mean, and on the slope of a ray from the origin to the corresponding point on the Lorenz curve of log-income in $t - 1$.

The shape of the TIM curve therefore clearly reflects the nature of the Galtonian process of income change. This specification also indicates that care needs to be taken in using the term ‘equalising’ when referring to the mobility process, since a substantial degree of regression ($\beta < 1$) can nevertheless be associated with an increasing variance of logarithms of income from $t - 1$ to $t$, depending on the extent of random variations, reflected in the variance, $\sigma_u^2$, of the $u_1$s. Where $\sigma_t^2$ is the variance of log-income at $t$, it is seen from (A.3) that:

$$ \sigma_t^2 = \beta^2 \sigma_{t-1}^2 + \sigma_u^2 $$

(A.11)

Hence, inequality in periods $t - 1$ and $t$ are equal only if $\sigma_u^2 = \sigma_t^2 (1 - \beta^2)$. It is possible to have $\sigma_{t-1}^2 > \sigma_t^2$ and a combination of $\beta$ and $\sigma_u^2$ such that the sum of incomes over the two periods is more equal than in both of the individual years. Alternatively, annual inequality can increase and ‘longer period’ inequality can lie between the values for the two separate years. Hence a concave TIM curve, reflecting substantial regression – with systematically higher income growth rates for lower-incomes – is not unambiguously associated with a reduction in annual inequality, depending on the extent of random variations (which are reflected in a less-smooth TIM curve). Longer-period inequality is less than the highest annual value, but is not necessarily lower than in all years.

**Appendix B: Alternative TIM and Income Share Curves**

This appendix provides additional information on the New Zealand income mobility data, and discusses an alternative TIM curve based on changes in the level of income, $dx$, as opposed to income growth rates, $d \ln(x)$. It then considers how the Fields (2000) notion of ‘income share mobility’ may be illustrated.

The empirical analyses throughout the paper are based on data from a 2 per cent random sample of individual New Zealand Inland Revenue personal income taxpayers, provided by the New Zealand Inland Revenue. It is not publicly available. The data cover three
longitudinal panels for the same 29,405 taxpayers over the five-year period 1998 to 2002 (4 years of income growth), 31,355 taxpayers for 2002-06 and 32,970 taxpayers for 2006-10. Each dataset was initially specified to eliminate taxpayers with less than $1,000 of income in the initial or terminal years.

The data for the 2006 to 2010 panel are illustrated in Figure 15, which shows the 2006 taxable income level of the individual at each ventile point (0.05, 0.10, 0.15, and so on) and the average income growth rate of individuals within each ventile group. This reveals, for example that income in the lowest ventile is around $6,500 and rises steadily to around $71,000 at the 18th ventile (the 90th percentile), thereafter rising more rapidly to around $175,000 in the top ventile (the value shown is income at the 99th percentile).

The regression to the mean feature of the data can be seen from the inverse pattern observed for income growth rates over 2006-10 in Figure 15, which generally decline at higher initial income levels. These income levels and growth rates have correlation, and rank correlation, coefficients of −0.52 and −0.97 respectively. The income growth rate is especially high in the lowest ventile at around 1.5 (or 150 per cent over the five years). However, this is unusual, with most ventile average growth rates ranging from around 0.2 to −0.2 (a 20 per cent change over five years). In total, nine ventiles display negative average
growth rates over the period, reflecting in part the impact of the global recession in 2008-10, such that many taxpayers’ 2010 income levels remained below their 2006 counterparts. The phenomenon of negative income growth for the highest ventiles is also observed, however, in the other two panels, 1998-2002 and 2002-06, again capturing systematic regression to the mean aspects, as observed for the three TIM curves in Figures 7 and 8.

**TIMs using Income Changes, $dx$**

Jenkins and Van Kerm (2016) examined Cumulative Income Growth Profiles (CIGPs), for income changes measured both as proportional income growth, $dy$, where $y = \ln x$, and in real income units, $dx$ (in January 2008 British £s). The TIM and normalised TIM curves presented in Section 5 are based on $dy$, but may readily be based on changes in income units. Figure 16 shows two normalised TIM curves obtained using the 2006-2010 income data: the lower panel is the equivalent of Figure 8 with $dx$ replacing $dy$. The upper panel has been normalised by subtracting, rather than dividing by, average income change, $\overline{dx}$. This helps to clarify the concave and convex ranges within the curve, which are less evident in the lower panel. Hence the upper curve ends at zero where the cumulative change per capita equals $\overline{dx}$.\(^{23}\)

Comparing Figure 8 with the lower panel of Figure 16 reveals that the latter is much closer to linear than the normalised TIM based on $dy$ between around the 10\(^{th}\) to the 95\(^{th}\) percentiles. This suggests that much of the observed concavity, indicating highly equalising mobility across the whole sample, is substantially due to similar absolute changes in income, $dx$, across much of the income distribution, translating into much higher income growth, $dy$, at lower income levels compared with higher income levels, except for the very highest and lowest incomes. The top panel of Figure 16 reveals this pattern to be less true when the details of income changes (in £s) can be seen. Nevertheless, the profile shows that, except at the lowest and highest incomes, in general income change per capita only varies by around $800 to $1,300 (relative to the average change of approximately $6,500) over 2006-10.\(^{24}\)

As with the TIM based on $dy$, the extent of equalising mobility is strongly influenced by income changes experienced by those on initially lowest (10 per cent), and highest (5 per cent), incomes.

\(^{23}\)The curve also starts at zero. However this is data-specific since the lowest income ranked individual in this case experienced no income change, hence $dx = 0$.

\(^{24}\)The values shown are presented in nominal, rather than real, dollar terms, since these data are not being compared with other years.
Figure 16: Normalised TIM Curves Based on Income Change (Instead of Proportional Income Change): 2006-2010
Profiles of Income Share Changes

One of the mobility concepts highlighted by Fields (2000) was *income share movement*. He suggested (p. 9) that, ‘[t]o the extent that people are relativists in their thinking, what they are much more likely to care about is their income as it compares with that of others. If your income rises by 50 percent but everyone else’s rises by 100 percent, you may feel that you have lost ground. Share-movement measures would say that you have experienced downward income mobility, precisely because your share of the total has fallen’.

Though Fields does not suggest a diagrammatic representation of share movement, changes in cumulative income shares over any two periods can be illustrated using Lorenz and concentration curves for incomes in two period ranked by first period income. Such curves are more usually presented for comparisons of two income cross-sections, though they can be constructed similarly from longitudinal data. Figure 17 shows the equivalent of the difference between such longitudinal-based Lorenz and concentration curves, but using the cumulative change in income share on the vertical axis and cumulative percentiles on the horizontal axis.

As with the normalised TIM curves in Figure 8, these profiles indicate a noticeably greater concavity for the 1998-2002 period than for the two later periods. All three profiles suggest that the income shares of the poorest 20 per cent of the sample experienced an
income share gain, on average, of around 0.05 (5 per cent) over each five-year period. The lowest half of the sample experienced around a 0.06 to 0.08 share increase, depending on the time period. Clearly, above-median individuals experienced the same decrease in income share. Since the three profiles refer to different samples of individuals, and hence are conditional on different initial income distributions, these results cannot simply be aggregated to identify longitudinal income changes over 1998-2010. Nevertheless, they appear to suggest quite significant income share increases over an extended period for individuals initially below median income.
References


