Dynamic Consistency and Regret

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Abstract

Individuals often report that they regret not having saved more for retirement. While many have interpreted such evidence as a rationale for modifying the discount function that applies to future utility (as in the case of hyperbolic discounting), we propose that another possible way to interpret the evidence is to just simply modify the discount function that applies to past utility. To make this point, we modify the standard discounted utility model as follows: we solve and simulate a “full control rights” life-cycle consumption and saving problem in which an individual at a given age, in the interior of the life cycle, has full control over all lifetime resources, including past, present, and future income. The forward discount function is exponential, which preserves the dynamic consistency of choices; but, by allowing for past utility to be discounted relative to present utility, the model can generate regret about not having saved more. For example, from the perspective of a 65 year old who is at retirement, the ideal level of assets to be holding can be twice as large as what is actually in hand, even though actual saving decisions are dynamically consistent across the entire life cycle. Adding an incomplete labor market with choice over retirement timing compounds the regret that individuals experience about past saving decisions.

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1 Introduction

In a variety of surveys, individuals report that they are not saving enough for retirement. For example, based on a random sample of 1,200 non-retired Americans, Farkus and Johnson (1997) report that 3 out of 4 respondents believe they should be doing more to set aside money for retirement, and 2 out of 5 Baby Boomers (who were 33-50 years old at the time of the survey) admit to having less than $10,000 set aside for retirement. Venti and Wise (2000) report that 3 out of 4 individuals in an HRS sub-sample say they have saved too little over the past 20 to 30 years. Similarly, 4 out of 5 workers in a random sample of Americans who participated in the 2016 Retirement Confidence Survey say they are “not very confident” that they have saved enough to finance a comfortable retirement, and about 1 out of 3 workers report that they are not saving anything at all.

Introduced by Samuelson (1937), the discounted utility model with exponential discounting has served as the benchmark template used by economists to examine intertemporal choice within the neoclassical tradition. The discounted utility model with exponential discounting is not compatible with self-reported data on regret about not saving enough for retirement. For example, Akerlof (2002, p.422) summarizes this point in his Nobel Prize acceptance speech:

For New Classical economics, saving too little or too much...is an impossibility, a straightforward contradiction of the assumptions of the model. Since saving is the result of individual utility maximization, it must, absent externalities, be just right...The hyperbolic discount function, which has been used to study intertemporal savings choices, can be used to formalize the distinction between the utility function that describes actual saving behavior and the utility function that measures the welfare resulting from that behavior.\(^1\)

\(^1\)Similarly, Rabin (1998, p.38) states:

An important qualitative feature of exponential discounting is that it implies that a person’s intertemporal preferences are time-consistent: A person feels the same about a given intertemporal tradeoff no matter when she is asked.
The presence of self-reported data on regret about saving too little for retirement (as well as related evidence from psychology) has motivated a number of economists to abandon the standard discounted utility model with exponential discounting in favor of behavioral theories like hyperbolic discounting. For instance, Camerer (1999, p.10577) states:

Behavioral economics can also provide a more realistic and thoughtful basis for making economic policy...For example, if people weight the future hyperbolically rather than exponentially, they will impulsively buy goods they will soon regret having bought.\(^2\)

While many have interpreted the evidence on self-reported undersaving as a rationale for modifying the discount function that applies to future utility (the forward discount function), we propose that another way to explain the evidence is to just simply modify the discount function that applies to past utility (the backward discount function). Of course, if the researcher truly wants a model in which an individual’s later choices deviate from what the initial self would like, then modifying the forward discount function—as in the case of hyperbolic discounting—is potentially justifiable.\(^3\) However, if a researcher simply wants to use a model that is compatible with the widespread observation that a person arrives at retirement with much less than he wishes that he had saved, then a simple modification to the backward discount function will do the trick.

In this paper, we modify the standard discounted utility model with exponential discounting, similar to Caplin and Leahy (2004) and İmrohoroğlu, İmrohoroğlu, and Joines (2003). We consider lifetime utility from different vantage points over the life cycle, and most importantly, we allow for past utility to be discounted relative to present utility. We note here that the underlying reason as to why individuals might discount past utility could be the widely documented empirical fact that people forget the past.\(^4\) We continue to assume exponential discounting of future utility as in the standard discounted utility model, which preserves the dynamic consistency of decision making.

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\(^2\) Similar statements to those above—that economists necessarily need to look to non-exponential discounting to explain undersaving and regret—can be found in Laibson (1997), Rabin (1998), O’Donoghue and Rabin (1999), and Frederick, Loewenstein, and O’Donoghue (2002), among many others.

\(^3\) If the researcher wants to further assume that future choices deviate from what the initial self is planning to do, then one must also assume individuals are naive rather than sophisticated (for example, see Prelec (2004)).

\(^4\) For evidence on memory decay, see Wixted and Ebbesen (1991), Rubin and Wenzel (1996), Wixted (2004), and Yi, Landes, and Bickel (2009), among many others.
over the life cycle. But, by allowing for past utility to be discounted relative to present utility, the model can generate regret about not having saved enough for retirement.

To make these points, we solve and simulate what we call a “full control rights” life-cycle consumption and saving problem in which an individual at a given age, in the interior of the life cycle, has full control over all lifetime resources, including past, present, and future income. We do not use the full control rights problem as the basis for understanding the individual’s actual choices; instead, we use it for the purpose of understanding, in a normative sense, the path that the individual wishes he could follow across the past, present, and future. In other words, the full control rights problem is an imaginary problem that does not govern actual choices, but instead it describes his ideal lifetime consumption and saving path from the perspective of the current vantage point. Even though the individual’s decision making is dynamically consistent, because of the exponential forward discount function, the solution to the full control rights problem is inconsistent with the decisions made by the individual under any discounting of past utility. The individual in the model makes a plan and sticks with it, and yet, we document that he regrets his past saving decisions for any degree of discounting of past utility.

For example, in one of the parameterizations of our simulated model, a 65 year old who discounts the future, but values the past the same as the present, would prefer to have accumulated 2 times more assets by retirement than he actually does accumulate. Likewise, if the past is discounted at the same rate as the future, then he would prefer to have accumulated 3 times more assets than what he does.

The regret experienced at age 65 is just one vantage point at which regret is experienced. At other ages, the individual prefers a different level of ideal asset accumulation, and thus experiences regret at all other ages in addition, relative to what is actually accumulated by that age. If welfare is defined by aggregating lifetime utility across each vantage point of the life cycle, as in Caplin and Leahy (2004) and İmrohoroğlu, İmrohoroğlu, and Joines (2003), then the optimal level of retirement savings can exceed the actual level by a factor of 1.6 if the individual values the past the same as the present, and by a factor of 2.2 if the individual discounts the past at the same rate as the future.

We focus particular attention on the role of the retirement decision itself in our analysis. While
the numbers reported above come from a life-cycle consumption and saving setting with exogenous retirement, we extend our analysis to include an incomplete labor market in which the individual works full time and then selects his ideal age of retirement. We find that the presence of retirement choice acts to compound the regret about one’s past saving decisions, because individuals now wish that they had saved more for two reasons: first, individuals wish they had saved more in the past so that they can consume more now in the present and in the future; and second, they wish that they had saved more in the past so that they could have financed an earlier age of retirement, compared to when they actually do retire based on their actual level of accumulated assets. Quantitatively, the regret about saving too little in the past is amplified in the presence of retirement choice, compared to a model environment in which labor supply is exogenous.

Our findings have implications for model selection in general. If the goal is to construct a model that is able to simply account for regret about saving too little for retirement, then modifying the backward discount function, in an otherwise standard discounted utility model, satisfies Occam’s razor for two reasons: First, the modified discounted utility model is observationally equivalent to the standard discounted utility model from the perspective that it does not disrupt the model’s predictions about actual behavior. Second, modifying the backward discount function is convenient given that it does not affect the dynamic consistency property regarding actual choice, and therefore, it allows the researcher to avoid the computational burden inherent in working with models of dynamically inconsistent choice.5

What are the policy implications of augmenting the standard discounted utility model as we have done? Thinking of a single individual as a collection of distinctly different time-dated selves, each with his own idea about ideal resource allocation (Pareto allocation) over the life cycle, we can speak of the welfare theorems. The first theorem holds because the actual choices made under Laissez Faire lead to a consumption-saving allocation that is Pareto optimal, meaning that the welfare of

5We are not suggesting that there is no evidence for hyperbolic discounting, nor are we making a statement about its role in economic theory. Indeed, Ameriks, Caplin, Leahy, and Tyler (2007) provide direct evidence on the existence of self-control problems among a segment of the population, a finding that cannot be explained by the standard discounted utility model. They also find that those who have difficulty saving as much as they plan to save tend to accumulate significantly less for retirement than those without a self-control problem. This is exactly the type of economic evidence that is needed to motivate theories like hyperbolic discounting.
the time-zero self is maximized. But, the second theorem does not hold, because lump-sum transfers of resources cannot support any target Pareto allocation: in the absence of any market frictions (like limitations on credit), the individual will not adjust his actual consumption choices in response to such transfers (as long as the net present value of the transfers is zero), and therefore only the most impatient allocation from the Pareto set is attainable. The asymmetry in power among the selves, that arises from their temporal ordering, prevents the second welfare theorem from holding.\footnote{Of course, a more empirically relevant issue is whether lump-sum transfers are desirable when credit markets are imperfect. In this case, transfers can successfully distort consumption and saving allocations in a way that could please later selves. Imrohoroglu, Imrohoroglu, and Joines (2003) study this issue thoroughly and use Caplin and Leahy's framework to show that, despite conventional wisdom, individuals who discount the future exponentially can potentially benefit from mandatory saving through Social Security. Also see Andersen and Bhattacharya (2011) for related results.}

Our paper is closely related to Caplin and Leahy (2004), who offer a radical critique of revealed preference welfare analysis in dynamic settings. At least since Samuelson (1938, 1953) and Houthakker (1950), economists have relied on the principle of revealed preference as a guide in conducting normative economic analysis. In this tradition, different policy proposals are evaluated through the lens of whatever utility function is used by individuals in their private decision making. For instance, if individuals discount the future at rate $x$ in their private choices, then so should the government in its evaluation of public projects that involve intertemporal tradeoffs. However, Caplin and Leahy caution that this tradition “rests on faith, not logic” (p.1257).

Recall from Strotz (1956) that choices in dynamic settings are consistent only if future utility is discounted exponentially. But since Strotz’s seminal work it seems to have gone almost unnoticed (until Caplin and Leahy) that, while exponential discounting of future utility leads to dynamically consistent choices, such choices do not necessarily reveal the preferences of all selves over the allocation of total lifetime resources. For example, if a 65 year old individual had full control over lifetime resources, then he may choose a different lifetime consumption allocation than what was chosen by his 18 year old self—an allocation that involves more saving for retirement—even though the 65 year old plans to stick to the original consumption-saving program going forward. In other words, if the assets of the 65 year old were not constrained by past saving decisions, he would typically opt to re-write those past decisions. In fact, the 65 year old would view the original plan...
as a suboptimal use of lifetime resources, except in the special case in which the 65 year old values
the past exponentially more than he values the present. Of course, in reality the 18 year old self
moves before the 65 year old self in the game of life, and the older self must take as given the assets
that he inherits from his younger selves. But Caplin and Leahy’s point is that revealed preference
welfare analysis arbitrarily favors the preferences of the 18 year old self. They question whether
policy makers really should discount future costs and benefits as impatiently as does an 18 year old.
If revealed preference is the basis for policy evaluation, then policy makers need only consult with
18 year olds before making decisions.

Caplin and Leahy’s paper forces economists to take a stand on how we treat the past in the
calculation of welfare. While conventional welfare analysis seems to be silent on this issue, in fact it
is not. By using the revealed preference welfare criterion, researchers are implicitly assuming that
all time-dated selves of a single individual agree on the ideal lifetime allocation. However, such
harmony of views is achieved only if individuals care exponentially more about the past than the
present. The revealed preference welfare criterion does not allow the researcher to side step the
seemingly messy issue of how to weight past experiences; instead, it forces the researcher to make
the unnatural (and, we suspect, empirically empty) assumption that the far distant past matters
more to the individual than anything else.

2 Four Optimization Problems

We summarize four different life-cycle optimization problems. The first is the canonical problem
of maximizing lifetime utility from the perspective of time 0. The second problem is the Strotz
(1956) problem of maximizing lifetime utility from the perspective of some older age, taking all

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7 Technically speaking, Caplin and Leahy make this point by considering the ideal lifetime consumption-saving
allocations of each time-dated self of a single individual, under the assumption that each self has full control rights
over past, present, and future resources. Each such solution represents a Pareto optimum, or an element from the
Pareto set. They then show that revealed preference welfare analysis is based (arbitrarily) on the most impatient
consumption-saving allocation from this Pareto set. The Pareto optima in the Pareto set are all identical only in
the special, knife-edge case where the individual cares exponentially more about the past than the present. For all
other cases in which past consumption is (weakly) discounted relative to present consumption—anywhere between
no discounting to infinite discounting—later selves will wish that earlier selves had saved more even though actual,
forward-looking decision making is dynamically consistent.
past choices as fixed. Strotz showed that the first and second problems share the same solution under exponential discounting. The third problem follows the tradition of Caplin and Leahy (2004) and is distinctly different from the other two problems: the individual maximizes lifetime utility from the perspective of some interior age, while treating all past, present, and future choices as free. We refer to this as the full control rights problem because the individual does not face any constraints beyond his own lifetime resource constraint. Finally, the fourth problem also follows the tradition of Caplin and Leahy in which a social planner maximizes aggregate lifetime utility, where the aggregation is over the lifetime utilities of the individual from every possible vantage point within the life cycle. In each of these four optimization problems, we purposefully focus on a simple, finite-horizon setting with no uncertainty about income, returns, or longevity, so that the possible existence of regret is not simply due to bad realizations from some stochastic process, but is instead a core result of discounting past utility (a simple modification to the standard discounted utility model).

Concerning notation: \( c(t) \) is consumption, \( k(t) \) is savings, \( r \) is the interest rate on savings, \( y(t) \) is disposable income, and \( T \) is the life span. The forward-looking (prospective) discount function for a delay of length \( \tau \) is \( F(\tau) \) and the backward-looking (retrospective) discount function is \( B(\tau) \). Note that we use the term “delay” to mean the absolute value of the length of time between the current moment and some other moment, whether that other moment is in the future or in the past. We set \( F(\tau) = e^{-\rho \tau} \) and \( B(\tau) = e^{-\rho \tau} \). Period utility is of the isoelastic variety, \( u(c) = c^{1-\sigma}/(1-\sigma) \).

### 2.1 Problem 1: Canonical Life-Cycle Consumption/Saving Problem

Standing at time zero, the individual solves

\[
\max : \int_0^T e^{-\rho \tau} c(t)^{1-\sigma} \frac{1}{1-\sigma} dt, \tag{1}
\]

subject to

\[
\frac{dk(t)}{dt} = rk(t) + y(t) - c(t), \tag{2}
\]

\[
k(0) = k(T) = 0. \tag{3}
\]
Using the Maximum Principle it is straightforward to show that the solution to this problem for \( t \in [0, \bar{T}] \) is

\[
c^*_0(t) = \frac{\int_0^\bar{T} y(t)e^{-rt}dt}{\int_0^\bar{T} e^{-\rho_F t/(1-\sigma)}e^{(r-\rho_F)t/\sigma}dt}, \quad k^*_0(t) = \int_0^t [y(s) - c^*_0(s)]e^{r(t-s)}ds. \tag{4}
\]

### 2.2 Problem 2: Dynamic Consistency (á la Strotz (1956))

Suppose the individual has followed the initial plan from time zero up to some point \( v \). If he were to re-optimize at vantage point \( v \), he would behave according to

\[
\text{max} : \int_v^\bar{T} e^{-\rho_F(t-v)}c(t)^{1-\sigma}dt, \tag{5}
\]

subject to

\[
\frac{dk(t)}{dt} = rk(t) + y(t) - c(t), \quad k(v) = \int_v^\bar{T} [y(t) - c^*_0(t)]e^{r(t-v)}dt, \quad k(\bar{T}) = 0. \tag{6}
\]

Again, with the Maximum Principle and some algebra it is straightforward to show (as in Strotz (1956)) that for \( t \in [v, \bar{T}] \)

\[
c^*_v(t) = c^*_0(t), \quad k^*_v(t) = k^*_0(t). \tag{8}
\]

### 2.3 Problem 3: Full Control Rights (á la Caplin and Leahy (2004))

Here, we first note that maximizing the objective functional of Problem 1 yields the same solution as the following,

\[
\text{max} : \int_0^\bar{T} e^{\rho_F(t-v)}c(t)^{1-\sigma}dt + \int_v^\bar{T} e^{-\rho_F(t-v)}c(t)^{1-\sigma}dt, \tag{1'}
\]

given that \((1')\) is just a scalar multiple of \((1)\). Even though this point has received little or no attention in the literature, this means that the canonical problem possesses the assumption that individuals retrospectively value past allocations at the negative of the forward-looking discount rate for positive discounting delays, meaning that individuals value the past more than the present,
aside from valuing the present more than the future. In what follows, we allow for the possibility that the backward discount rate can take on a value that is different than the negative of the forward discount rate.

Suppose the individual is standing at age $v$, but unlike in Problem 2, he calculates the entire life-cycle consumption-saving program that he views as optimal from the perspective of age $v$. That is, at age $v$ he ignores the reality of his current asset balance, inherited from himself through past saving decisions. Instead, he imagines what could have been: he computes the ideal plan for his past, present, and future as if he has full control rights over all lifetime resources. This program is the solution to a two-stage control problem

$$\max : \int_{0}^{v} e^{-\rho_{B}(v-t)} \frac{c(t)^{1-\sigma}}{1-\sigma} dt + \int_{v}^{T} e^{-\rho_{F}(t-v)} \frac{c(t)^{1-\sigma}}{1-\sigma} dt,$$  \tag{9}$$

subject to

$$\frac{dk(t)}{dt} = rk(t) + y(t) - c(t), \tag{10}$$

$$k(0) = k(T) = 0. \tag{11}$$

In Appendix A, we show that the optimal consumption path is

$$c_v^{**}(t) = \frac{\int_{0}^{T} y(s)e^{-rs}ds}{\int_{0}^{v} [e^{\rho_{B}(v-s)-rs}]^{-1/\sigma} e^{-rs} ds + \int_{v}^{T} [e^{\rho_{F}(s-v)-rs}]^{-1/\sigma} e^{-rs} ds} \left[ e^{\rho_{B}(v-t)-rt} \right]^{-1/\sigma}, \tag{12}$$

looking backward over $t \in [0, v]$, and looking ahead over $t \in [v, T]$ the optimal consumption path is

$$c_v^{**}(t) = \frac{\int_{0}^{T} y(s)e^{-rs}ds}{\int_{0}^{v} [e^{\rho_{B}(v-s)-rs}]^{-1/\sigma} e^{-rs} ds + \int_{v}^{T} [e^{\rho_{F}(s-v)-rs}]^{-1/\sigma} e^{-rs} ds} \left[ e^{\rho_{F}(t-v)-rt} \right]^{-1/\sigma}. \tag{13}$$

We emphasize that $c_v^{**}(t) = c_v^{*}(t)$ and $k_v^{**}(t) = k_v^{*}(t)$ only in the knife-edge case where $-\rho_{B} = \rho_{F}$, as can be anticipated by a careful comparison of (1’) with (9). If $-\rho_{B} \neq \rho_{F}$, then such a simple modification to the standard model will lead to the presence of regret about past decisions. In what follows, we consider parameterizations in which individuals discount the past, instead of placing exponentially more weight on the past than the present, as is the case in the standard model.
Numerical examples illustrate the quantitative magnitude of the regret that the individual will experience from vantage point $v$. We set $T = 55$ to reflect an economic life span from ages 25 to 80. Assuming retirement occurs after 40 years of work, we set $v = 40$ to capture the preferences of an individual at the date of retirement. We assume $y(t) = 1$ before 40 and $y(t) = 0.4$ afterwards to reflect social security benefits and other sources of income such as part-time work. We set $r = 1\%$ to align with typical risk-free returns on US treasuries and we set $\sigma = 1$. Finally, we assume a modest amount of forward discounting, $\rho_F = 2\%$.

All that remains is to select the backward discount rate $\rho_B$. We consider four parameterizations that vary in the value that we assign to $\rho_B$. In each parameterization, the individual follows a dynamically-consistent consumption-saving program in the sense that his actual choices always follow his initial plan. At a knife-edge parameterization (Parameterization 1 below), the individual does not regret his past choices. Yet, at all other parameterizations (e.g., Parameterizations 2-4 below), he regrets his past choices and wishes that he had saved more over the interval $[0, v]$.

**Parameterization 1:** $\rho_B = -\rho_F$. For this knife-edge case, Problems 1 and 3 are mathematically equivalent and there is no regret about past choices. But notice that this requires a strong, perhaps counterfactual, assumption that an individual cares exponentially more about the past than the present.

**Parameterization 2:** $\rho_B = \rho_F$. Here, the present is salient and the individual discounts both the future and the past at the same rate over all delays, whether forward or backward. The memory of a great vacation one year ago provides the same utility as the expectation of a great vacation one year from now. This is our preferred parameterization, given that discounting of the past is consistent with the widely documented empirical fact that people forget the past.

**Parameterization 3:** $\rho_B = 0$. The individual cares just as much about each point in the past as he cares about the present.

**Parameterization 4:** $\rho_B = \infty$. The individual derives no utility whatsoever from past consumption and only cares about the present and future.
Figure 1 illustrates how a 65-year-old on the verge of retirement would value allocations over time. He discounts the future according to the forward discount function $e^{-\rho_F(t-v)}$, while he discounts past consumption according to the backward discount function $e^{-\rho_B(v-t)}$. We have plotted four different possibilities for the backward discount function (referred to as Parameterizations 1-4 above).

In Figure 2 we plot the optimal consumption path $c_v^{**}(t)$ from the vantage point of retirement (i.e., $v$ is set to model time 40, or age 65). This is the path that the individual would choose for his entire life cycle, if he had full control over his lifetime resources. Of course, he does not have full control; he is stuck with the endowment left by previous selves. Because he discounts the future exponentially, he will indeed stick with his initial consumption plan $c_0^v(t)$ no matter how he discounts the past. But notice how much differently he would like to have consumed if he could rewrite the past. Only in the knife-edge (counterfactual) case in which he cares exponentially more about the past than the present would he ever agree with the decisions of his past selves.

Figure 3 is similar to Figure 2, but now we plot optimal savings $k_v^{**}(t)$ from the vantage point of an individual at retirement. Again, only in the knife-edge case in which the individual cares exponentially more about the past than the present would he feel that he had saved just the right amount for retirement. In this case, $k_v^{**}(t) = k_0^v(t)$. But for the other, more plausible parameterizations in which the individual cares less about the past than the present ($\rho_B \geq 0$), he will experience significant regret. For instance, in our preferred parameterization where the individual discounts the past at the same rate that he discounts the future ($\rho_B = \rho_F$), he will wish that he had saved three times more for retirement than what he actually does save. In fact, as long as $\rho_B > -\rho_F$, the individual will regret having saved “too little.” Even in the case where the individual does not discount the past at all ($\rho_B = 0$), he will still wish that he had saved and accumulated about twice as much for retirement compared to what he actually does accumulate. These findings document how such a simple modification to the standard discounted utility model can deliver predictions that are consistent with data on regret about saving too little for retirement, while preserving the convenient feature of the standard model that actual choice is dynamically consistent.
2.4 Problem 4: Welfare (á la Caplin and Leahy (2004))

The solution consumption-saving allocation from the previous problem is optimal from the perspective of age \( v \), but not from any other age. Hence, there is disagreement among the selves concerning how allocations should be valued. Each solution represents a Pareto optimum, or a cross-section of the Pareto surface.

Utility from the perspective of the individual standing at age \( v \) is

\[
U(v) \equiv \int_0^v e^{-\rho_B(v-t)}u(c(t))dt + \int_v^T e^{-\rho_F(t-v)}u(c(t))dt. \tag{14}
\]

“Social welfare” respects the preferences of all the different selves,

\[
SW \equiv \int_0^T \alpha(v)U(v)dv, \tag{15}
\]

where we set the Pareto weights to \( \alpha(v) = 1 \) in what follows. Thus, the socially optimal consumption profile solves the following control problem

\[
\max : \int_0^T \left( \int_0^v e^{-\rho_B(v-t)}u(c(t))dt + \int_v^T e^{-\rho_F(t-v)}u(c(t))dt \right) dv, \tag{16}
\]

subject to

\[
\frac{dk(t)}{dt} = rk(t) + y(t) - c(t), \tag{17}
\]

\[
k(0) = k(T) = 0. \tag{18}
\]

In Appendix B, we rewrite the objective compactly

\[
\max : \int_0^T D(t)u(c(t))dt, \tag{19}
\]

where \( D(t) \) is the social discount function

\[
D(t) \equiv \frac{1}{\rho_B} \left( 1 - e^{-\rho_B(T-t)} \right) + \frac{1}{\rho_F} \left( 1 - e^{-\rho_F t} \right). \tag{20}
\]
The solution to this optimal control problem for isoelastic utility is

$$c^*_{SW}(t) = \frac{\int_0^T y(t)e^{-rt}dt}{\int_0^T e^{rt/\sigma} D(t)^{1/\sigma} e^{-rt} dt} e^{rt/\sigma} D(t)^{1/\sigma},$$

(21)

$$k^*_{SW}(t) = \int_0^t [y(s) - c^*_{SW}(s)] e^{r(t-s)} ds.$$  

(22)

As in the previous subsection, we continue to use the same four parameterizations of the backward discount rate $\rho_B$ in numerical examples. In Appendix C, we document the properties of the social discount function for each of these four parameterizations. Figure 4 plots the social discount function $D(t)$ that corresponds to each of the alternative parameterizations of the backward discount function. Each social discount function has been normalized to ensure that it peaks at unity. Figure 5 plots socially optimal consumption $c^*_{SW}(t)$ for each of these cases, and Figure 6 reports the socially optimal savings asset profiles that result $k^*_{SW}(t)$. In our preferred parameterization ($\rho_B = \rho_F$), the social optimum requires twice as much savings by the date of retirement compared to what the individual actually accumulates on his own. Even if the individual cares just as much about the past as he cares about the present ($\rho_B = 0$), and therefore prefers relatively high levels of consumption when young, it is still the case that the individual seriously undersaves for retirement. Finally, if the individual does not care at all about the past ($\rho_B = \infty$) then undersaving is the most pronounced.

3 Choice over Retirement Timing

In this section we extend our analysis to include the decision of when to retire. We show that the retirement margin of choice has a first-order impact on the degree of regret that individuals feel about their past saving behavior. Regret about past saving decisions is compounded in our model because individuals wish that they had saved more in the past for two reasons: first, so that the present self and future selves can enjoy higher levels of consumption; and second, to finance an earlier age of retirement. When we decompose the disagreement about retirement savings among
the time-dated selves into the part that would occur if retirement is fixed and the overall effect
when retirement timing is a choice, we find that the second effect is much larger than the first. In
other words, the regret about past saving decisions is significantly amplified by the presence of the
retirement margin of choice.

We solve and simulate a life-cycle consumption and saving model such that the individual chooses
to actually retire from the labor force at age 66, in the presence of an incomplete labor market. This
is the optimal retirement age from the perspective of age 25 (model time zero), and it is the age at
which he ultimately does retire given the dynamic consistency property governing actual choices, a
result of having an exponential forward discount function. However, as the individual ages, he will
begin to wish that he had saved more when younger and he would like to retire earlier than planned,
even though his decision making will continue to follow the original consumption and saving path
and he will ultimately retire exactly as initially planned. For example, when the individual arrives
at the age of 40, he will almost be finished with repaying his debts and he will be ready to begin
saving for retirement. Yet he will wish that he had never borrowed anything at all in the past, and
instead, he will regret that he is not further ahead in being able to finance an earlier retirement.

3.1 Notation

Recalling our previous notation, $t$ is time, $c(t)$ is consumption, and all saving is done in a risk-free
bond account, $k(t)$, where $r = 0$ for convenience. Let $y(t)$ be disposable income, $T$ is retirement,
and $\bar{T}$ is the certain life span. Following Heijdra and Romp (2009) and Dybvig and Liu (2010), the
nature of the labor market is such that an individual starts the economic life cycle as a worker,
labor is indivisible during the working years, and retirement is a one-time, irreversible event that
causes a permanent switch from full-time work to no earned income. Wage income is normalized
to $y(t) = 1$ when working, period utility from consumption is $u(c)$, and the additively separable
fixed utility cost of work each period is $\psi$. Hence, the individual’s period utility is $u(c) - \psi$ when
working and $u(c)$ when retired.
3.2 Full Control Rights over Consumption and Labor

Suppose the individual is standing at vantage point \( v \in [0, \bar{T}] \), but unlike a standard problem in which he takes existing assets at age \( v \) as given, he now imagines the entire life-cycle consumption-saving program that he views as optimal. That is, he ignores the reality of how much (or little) he has already accumulated in his asset account by age \( v \), and instead imagines what might have been: he imagines the ideal program for his past, present, and future consumption, saving, and labor decisions. This program is the solution to a multi-stage optimal control problem with full control rights:

\[
\max_{\{c(t),T\}} \int_0^v B(v - t)[u(c(t))] - 1\{t \leq T\} \psi]dt + \int_v^{\bar{T}} F(t - v)[u(c(t))] - 1\{t \leq T\} \psi]dt, \tag{23}
\]

subject to

\[
\frac{dk(t)}{dt} = 1\{t \leq T\} - c(t), \tag{24}
\]

\[
k(0) = k(\bar{T}) = 0. \tag{25}
\]

We break this problem into an inner problem and an outer problem. In the inner problem we solve for the optimal consumption and saving program for a fixed age of retirement. In the outer problem we solve for the optimal age of retirement, given the optimal consumption and saving program. The value function in the outer problem is not always concave and does not always admit interior solutions so we optimize by brute force (we try every \( T \) on the computer and keep the argmax). We document the details of the solution procedure in Appendix D.

We denote the optimal age of retirement from the perspective of vantage point \( v \) as \( T^*(v) \). Likewise, optimal consumption and savings at age \( t \), from the perspective of vantage point \( v \) and conditional on retirement occurring at the optimal date \( T^*(v) \), are denoted \( c^*(t|T^*(v), v) \) and \( k^*(t|T^*(v), v) \).
3.3 Numerical Examples

We consider standard, dynamically-consistent decision making as a result of $F(\tau) = e^{-\rho_F \tau}$. We also assume the individual discounts past utility exponentially, though not necessarily at the same rate that he discounts future consumption, $B(\tau) = e^{-\rho_B \tau}$. Let period utility from consumption be $u(c) = c^{1-\sigma}/(1 - \sigma)$. Under these assumptions, optimal consumption and retirement decisions from vantage point $v$ can be summarized succinctly as

$$T^*(v) = \arg \max_T \left[ \frac{T}{1-\sigma} \lambda - \psi \left( \int_0^{\min\{v,T\}} e^{-\rho_B (v-t)} dt + \int_v^{\max\{v,T\}} e^{-\rho_F (t-v)} dt \right) \right], \quad (26)$$

$$\lambda \equiv \left( \frac{T}{\int_0^v e^{-\rho_B (v-s)/\sigma} ds + \int_v^T e^{-\rho_F (s-v)/\sigma} ds} \right)^{-\sigma}, \quad (27)$$

$$c^*(t|T^*(v), v) = \frac{T^*(v)}{\int_0^v e^{-\rho_B (v-s)/\sigma} ds + \int_v^T e^{-\rho_F (s-v)/\sigma} ds} \times \begin{cases} e^{-\rho_B (v-t)/\sigma}, & \text{for } t \in [0, v], \\ e^{-\rho_F (t-v)/\sigma}, & \text{for } t \in [v, T]. \end{cases} \quad (28)$$

Our baseline parameter values are: $\rho_F = 4\%$, $\sigma = 2$, $\bar{T} = 55$ (to reflect an economic life span from ages 25 to 80). Given these values, we set $\psi = 3.412$ to generate an actual selected age of retirement at age 66 (model age 41). That is, $T^*(0) = 41$ when $\psi = 3.412$.

Notice that we do not need to make any assumption about the backward discount rate $\rho_B$ when calibrating the parameter $\psi$ because the backward rate is irrelevant to the actual dynamically-consistent consumption, saving, and retirement decisions of the individual. And, regardless of the backward rate, as the individual ages he will stick with his initial choices because he discounts future utility exponentially. However, the backward rate is relevant when computing the ideal consumption, saving, and retirement decisions of the later selves, when these later selves are hypothetically granted full control rights over lifetime resources. We again consider three parameterizations of $\rho_B$ outlined above:

**Parameterization 2:** $\rho_B = \rho_F$. Past utility is discounted the same as future utility.
Parameterization 3: \( \rho_B = 0 \). Past utility is not discounted.

Parameterization 4: \( \rho_B = \infty \). The past provides no current utility.

Figure 7 shows the optimal retirement age \( T^*(v) \) as a function of vantage point \( v \), for all three parameterizations of \( \rho_B \). We plot two reference lines: first, the retirement date that is optimal from the initial vantage point \( T^*(0) \), which is also the retirement date that is actually experienced (because the forward discount function is exponential, which yields dynamically-consistent choice); and second, a 45 degree line to easily observe the relation between the ideal age of retirement and the current vantage point. Notice that the individual initially wants to retire at age 66, and all three \( T^*(v) \) profiles trivially equal 66 when \( v = 0 \), because no time has yet elapsed and therefore backward discounting is not yet relevant. All three \( T^*(v) \) profiles decline early on, reflecting the individual’s regret over not having saved more on the interval \([0, v]\) to finance and enjoy an earlier retirement. The rate at which \( T^*(v) \) drops is related to \( \rho_B \) in an intuitive way. The higher \( \rho_B \), the sharper the decline. For example, when \( \rho_B = \infty \) the individual experiences the sharpest regret as he ages because of not having saved more to finance a very early retirement. In fact, in this case, by the time the individual reaches age 40, he wishes that he had saved enough to retire at that very moment, even though he recognizes that he is on a savings path that will lead to retirement at age 66. As he ages beyond 40, he continues to wish that he were retiring at his current vantage point (and not a moment sooner, because he wants to subject past selves to working and saving every amount of past wage income).

For more reasonable assumptions, such as \( \rho_B = \rho_F \), the individual will eventually want to retire when he is 50, and as he ages beyond 50 he will believe it is ideal to have retired earlier than his current age. Thus, there is an interesting range of ages between 50 and 66 in which this individual will be working and will recognize that he is on a dynamically consistent consumption and saving program that will lead to retirement at age 66, and yet he will wish that he could be on a different program that would have resulted in him already being retired. Unlike the case of infinite backward discounting (\( \rho_B = \infty \)) in which \( T^*(v) \) rides the 45 degree line because those older selves always want younger selves to work, in this case the older selves do care about the consumption and leisure
experienced by their younger selves and hence do prefer that at least some of those selves could have enjoyed the utility that comes from retirement. Finally, the graph of $T^*(v)$ for the case of no backward discounting ($\rho_B = 0$) is similar to the case of $\rho_B = \rho_F$. Both graphs decline until they cross the 45 degree line and then rise gently thereafter. In all cases it is clear that disagreement among the many selves about the ideal retirement date is very pronounced.

In the next set of figures we plot savings profiles. We use the following notation. A savings profile or savings allocation $k^*(t|T, v)$ is the optimal balance in the savings account at age $t$, condition on retirement occurring at age $T$ and conditional on the individual’s current vantage point $v$. Figure 8 plots ideal savings allocations under full control rights. The allocation that is optimal from the initial vantage point $k^*(t|T(0), 0)$ is plotted as a reference. The other three graphs correspond to vantage point $v = 15$ (age 40), for the three different parameterizations of $\rho_B$. This figure conveys two important points. First, we can easily see how a 40 year old would disagree with his earlier 25 year old self about how much to save for retirement, and second, we can see how this disagreement depends on the particular assumptions that we make about backward discounting $\rho_B$.

Figures 9, 10, and 11 illustrate our main point. Each of these figures decomposes the disagreement about retirement savings among the selves into the part that would occur if retirement is fixed and the overall effect when retirement is endogenous. Like Figure 8, these figures continue to use vantage point $v = 15$ (age 40) as an example to illustrate the point. The figures differ only in the value that is assigned to $\rho_B$. The point of these figures is to show how the disagreement over how much should be in the savings account at the date of retirement is amplified by the presence of the retirement margin of choice. Consider the case of $\rho_B = \rho_F$ (Figure 9). If retirement is held exogenously fixed at age 66, then a 40 year old would wish that he was on a path that would ultimately involve 11% more savings by age 66 than the amount that will actually be realized. Alternatively, under endogenous retirement choice, a 40 year old would wish that he was on a very different path that would ultimately involve 66% more savings for retirement (given the earlier retirement age of 54, that is ideal from his current perspective), compared to the level of savings at the age of retirement that he will actually experience. Age 40 is just one possible vantage point to depict this result. But our point is that the disagreement among the various time-dated selves over how
much to save for retirement becomes dramatically amplified when the retirement margin of choice is operative in the model.

4 Concluding Remarks

Building on concepts developed by Caplin and Leahy (2004), we document that just a simple modification to the standard discounted utility model with exponential discounting can deliver predictions consistent with the idea (and consistent with self-reported data) that individuals regret not having saved more for retirement. We make this point by constructing, solving, and simulating a “full control rights” life-cycle consumption and saving model in which an individual at a given age, in the interior of the life cycle, has full control over all lifetime resources. The actual choices of the individual are dynamically consistent, a result of possessing an exponential forward discount function. But, by allowing for past utility to be discounted relative to present utility, the individual will experience regret about not having saved more for retirement as a result of his past choices. The full control rights problem does not dictate actual saving choices, but instead, it is an imaginary problem that characterizes the ideal lifetime consumption and saving allocation from the perspective of the current age of the individual. We find that the difference between what is actually saved, as a result of dynamically consistent choice, and what is ideal from the individual’s perspective of retirement is economically significant, meaning that the individual almost always experiences regret about not having saved more when younger. We extend our analysis to a life-cycle environment with choice over retirement timing in the presence of incomplete labor markets, and we find that the gap between actual and ideal savings for retirement is amplified dramatically. Our findings indicate that just a simple modification to the standard discounted utility model, such that both past utility and future utility are discounted, can deliver predictions that are consistent with a sizable body of evidence that individuals regret not having saved more for retirement.
Appendix A: Full Control Rights

Given the dynamic optimization problem outlined in (9)–(11), we form a pair of Hamiltonians

\[ H_1 = e^{-\rho_B(v-t)} \frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda_1(t)[rh(t) + y(t) - c(t)], \quad \text{for} \ t \in [0, v], \quad (A1) \]

\[ H_2 = e^{-\rho_F(t-v)} \frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda_2(t)[rh(t) + y(t) - c(t)], \quad \text{for} \ t \in [v, \bar{T}], \quad (A2) \]

following the Two-Stage Maximum Principle (Tomiyama (1985)). The first-order conditions include

\[ \frac{\partial H_1}{\partial c(t)} = e^{-\rho_B(v-t)} c(t)^{-\sigma} - \lambda_1(t) = 0, \quad \text{for} \ t \in [0, v], \quad (A3) \]

\[ \frac{\partial H_2}{\partial c(t)} = e^{-\rho_F(t-v)} c(t)^{-\sigma} - \lambda_2(t) = 0, \quad \text{for} \ t \in [v, \bar{T}], \quad (A4) \]

\[ \frac{d\lambda_1(t)}{dt} = -\frac{\partial H_1}{\partial k(t)} = -r \lambda_1(t), \quad \text{for} \ t \in [0, v], \quad (A5) \]

\[ \frac{d\lambda_2(t)}{dt} = -\frac{\partial H_2}{\partial k(t)} = -r \lambda_2(t), \quad \text{for} \ t \in [v, \bar{T}], \quad (A6) \]

\[ \lambda_1(v) = \lambda_2(v). \quad (A7) \]

Solving the costate equations gives

\[ \lambda_1(t) = a_1 e^{-rt}, \quad \text{for} \ t \in [0, v], \quad (A8) \]

\[ \lambda_2(t) = a_2 e^{-rt}, \quad \text{for} \ t \in [v, \bar{T}], \quad (A9) \]

for constants \( a_1 \) and \( a_2 \). The matching condition implies \( a_1 = a_2 \) and hence we can drop the subscripts

\[ \lambda(t) = ae^{-rt}, \quad \text{for} \ t \in [0, \bar{T}]. \quad (A10) \]
Rewrite the Maximum Conditions

\[ c(t) = [ae^{\rho_B(v-t)-rt}]^{-1/\sigma}, \text{ for } t \in [0, v], \]  \hspace{1cm} (A11)

\[ c(t) = [ae^{\rho_F(t-v)-rt}]^{-1/\sigma}, \text{ for } t \in [v, \bar{T}]. \]  \hspace{1cm} (A12)

From the state equation we have

\[
k(t) = \begin{cases} 
\int_0^t \{ y(s) - [ae^{\rho_B(v-s)\delta}]^{-1/\sigma} \} e^{r(t-s)} ds, & \text{for } t \in [0, v], \\
 k(v) e^{r(t-v)} + \int_v^t \{ y(s) - [ae^{\rho_F(s-v)\delta}]^{-1/\sigma} \} e^{r(t-s)} ds, & \text{for } t \in [v, \bar{T}].
\end{cases} \]  \hspace{1cm} (A13)

Evaluate \( k(t) \) at \( t = \bar{T} \) and use \( k(\bar{T}) = 0 \)

\[
0 = \int_0^v \left\{ y(s) - [ae^{\rho_B(v-s)\delta}]^{-1/\sigma} \right\} e^{r(v-s)} ds \times e^{r(\bar{T}-v)} + \int_v^\bar{T} \left\{ y(s) - [ae^{\rho_F(s-v)\delta}]^{-1/\sigma} \right\} e^{r(\bar{T}-s)} ds
\]

\[
= \int_0^v \left\{ y(s) - [ae^{\rho_B(v-s)\delta}]^{-1/\sigma} \right\} e^{-rs} ds + \int_v^\bar{T} \left\{ y(s) - [ae^{\rho_F(s-v)\delta}]^{-1/\sigma} \right\} e^{-rs} ds
\]  \hspace{1cm} (A14)

and then solve for \( a \)

\[
a^{-1/\sigma} = \frac{\int_0^\bar{T} y(s)e^{-rs} ds}{\int_0^v [e^{\rho_B(v-s)\delta}]^{-1/\sigma} e^{-rs} ds + \int_v^\bar{T} [e^{\rho_F(s-v)\delta}]^{-1/\sigma} e^{-rs} ds}. \]  \hspace{1cm} (A15)

Hence, looking backward over \( t \in [0, v] \) the optimal consumption path is

\[
c^*_c(t) = \frac{\int_0^\bar{T} y(s)e^{-rs} ds}{\int_0^v [e^{\rho_B(v-s)\delta}]^{-1/\sigma} e^{-rs} ds + \int_v^\bar{T} [e^{\rho_F(s-v)\delta}]^{-1/\sigma} e^{-rs} ds} [e^{\rho_B(v-t)\delta}]^{-1/\sigma}. \]  \hspace{1cm} (A16)
and looking ahead over \( t \in [v, \bar{T}] \) the optimal consumption path is

\[
c_v^*(t) = \frac{\int_0^\bar{T} y(s) e^{-rs} ds}{\int_0^v [e^{\rho_B(v-s)} - rs]^{-1/\sigma} e^{-rs} ds + \int_v^\bar{T} [e^{\rho_F(s-v)} - rs]^{-1/\sigma} e^{-rs} ds}
\]

(A17)

**Appendix B: Re-expression of Social Welfare Function**

Rewrite (16)

\[
\int_0^\bar{T} \int_0^v e^{-\rho_B(v-t)} u(c(t)) dt \, dv + \int_0^\bar{T} \int_v^\bar{T} e^{-\rho_F(t-v)} u(c(t)) dt \, dv = \int_0^\bar{T} e^{-\rho_B} \left( \int_0^v e^{\rho_B} u(c(t)) dt \right) dv + \int_0^\bar{T} e^{\rho_F} \left( \int_v^\bar{T} e^{\rho_F} u(c(t)) dt \right) dv.
\]

(B1)

Integrate by parts

\[
\int_0^\bar{T} e^{-\rho_B} \left( \int_0^v e^{\rho_B} u(c(t)) dt \right) dv = \left[ \frac{1}{-\rho_B} e^{-\rho_B} \left( \int_0^v e^{\rho_B} u(c(t)) dt \right) \right]_0^\bar{T} + \int_0^\bar{T} \frac{1}{\rho_B} u(c(v)) dv
\]

\[
= \frac{1}{-\rho_B} e^{-\rho_B} \left( \int_0^\bar{T} e^{\rho_B} u(c(t)) dt \right) + \int_0^\bar{T} \frac{1}{\rho_B} u(c(t)) dt
\]

\[
= \int_0^\bar{T} \frac{1}{\rho_B} (1 - e^{-\rho_B(T-t)}) u(c(t)) dt.
\]

(B2)

\[
\int_0^\bar{T} e^{\rho_F} \left( \int_v^\bar{T} e^{-\rho_F} u(c(t)) dt \right) dv = \left[ \frac{1}{\rho_F} e^{\rho_F} \left( \int_v^\bar{T} e^{-\rho_F} u(c(t)) dt \right) \right]_0^\bar{T} + \int_0^\bar{T} \frac{1}{\rho_F} u(c(v)) dv
\]

\[
= \frac{-1}{\rho_F} \left( \int_0^\bar{T} e^{\rho_F} u(c(t)) dt \right) + \int_0^\bar{T} \frac{1}{\rho_F} u(c(t)) dt
\]

\[
= \int_0^\bar{T} \frac{1}{\rho_F} (1 - e^{-\rho_F t}) u(c(t)) dt.
\]

(B3)

Thus, (16) can be re-stated compactly as (19) where

\[
D(t) \equiv \frac{1}{\rho_B} (1 - e^{-\rho_B(T-t)}) + \frac{1}{\rho_F} (1 - e^{-\rho_F t}),
\]

(B4)
is the social discount function that assigns weights to the different time-dated selves.

**Appendix C: Properties of the Social Discount Function**

For the parameterization, $\rho_B = -\rho_F$, the social discount function is the same as the private, forward discount function:

$$D(t) = \frac{1}{-\rho_F} \left( 1 - e^{\rho_F(T-t)} \right) + \frac{1}{\rho_F} \left( 1 - e^{-\rho_F t} \right)$$

$$= \frac{1}{\rho_F} \left[ e^{\rho_F T} - 1 \right] e^{-\rho_F t}$$

$$\propto e^{-\rho_F t}. \quad (C1)$$

At this knife-edge parameterization of the backward discount rate, the preferences of all the different selves are in perfect agreement concerning the valuation of allocations over time, and so the social discount function reflects this harmony.

For the parameterization, $\rho_B = \rho_F = \rho$, $D(t)$ is strictly concave (quadratic) with a peak at $t = T/2$:

$$D(t) = \frac{1}{\rho} \left[ 2 - e^{-\rho(T-t)} - e^{-\rho t} \right]. \quad (C2)$$

$$D'(t) = \frac{1}{\rho} \left[ -\rho e^{-\rho(T-t)} + \rho e^{-\rho t} \right]. \quad (C3)$$

$$D'(t) = 0 \iff t = \frac{T}{2}. \quad (C4)$$

$$D''(t) = \frac{1}{\rho} \left[ -\rho^2 e^{-\rho(T-t)} - \rho^2 e^{-\rho t} \right] < 0. \quad (C5)$$

This function has a quadratic shape because the midpoint is, on average, the closest to all the various vantage points and hence it is discounted the least in an aggregate sense. On the other hand, the boundaries of the life cycle are, on average, the furthest from all the vantage points and so they get the least weight in the social optimization problem.

For the parameterization, $\rho_B = 0$, $D(t)$ is strictly concave and strictly decreasing with a peak
at $t = 0$:

\[
\lim_{\rho_B \to 0} D(t) = \lim_{\rho_B \to 0} \frac{1}{\rho_B} (1 - e^{-\rho_B(T-t)}) + \lim_{\rho_B \to 0} \frac{1}{\rho_F} (1 - e^{-\rho_F t})
\]

\[
= \lim_{\rho_B \to 0} \frac{d}{d\rho_B} \left( \frac{1}{\rho_B} (1 - e^{-\rho_B(T-t)}) \right) + \frac{1}{\rho_F} (1 - e^{-\rho_F t})
\]

\[
= \lim_{\rho_B \to 0} \frac{(\bar{T} - t)}{1} e^{-\rho_B(T-t)} + \frac{1}{\rho_F} (1 - e^{-\rho_F t})
\]

\[
= \bar{T} - t + \frac{1}{\rho_F} (1 - e^{-\rho_F t}) .
\]

(C6)

\[
\frac{d}{dt} \left( \lim_{\rho_B \to 0} D(t) \right) = -1 + e^{-\rho_F t} < 0 .
\]

(C7)

\[
\frac{d^2}{dt^2} \left( \lim_{\rho_B \to 0} D(t) \right) = -\rho_F e^{-\rho_F t} < 0 .
\]

(C8)

Individuals have perfect recall in the sense that a fun vacation at any point in the past is just as valuable today as a fun vacation today.

For the parameterization, $\rho_B = \infty$, $D(t)$ is strictly concave and strictly increasing, with a peak at $t = \bar{T}$:

\[
D(t) = \frac{1}{\infty} (1 - e^{-\infty(T-t)}) + \frac{1}{\rho_F} (1 - e^{-\rho_F t})
\]

\[
= \frac{1}{\rho_F} (1 - e^{-\rho_F t}) .
\]

(C9)

\[
D'(t) = e^{-\rho_F t} > 0 .
\]

(C10)

\[
D''(t) = -\rho_F e^{-\rho_F t} < 0 .
\]

(C11)

In this case individuals do not care at all about the past, but all of the selves care about utility when old (to varying degrees). The social discount function respects this “consensus” opinion about the importance of old-age consumption and places the most weight on utility when old.
Appendix D: Full Control Rights with Retirement Choice

We can expand and rewrite the objective functional as

\[
\max_{\{c(t), T\}}: \int_0^v B(v - t)u(c(t))dt + \int_v^T F(t - v)u(c(t))dt - \int_0^{\min\{v,T\}} B(v - t)\psi dt - \int_v^{\max\{v,T\}} F(t - v)\psi dt. 
\]

(D1)

Notice that the upper limits on the third and fourth integrals in the objective functional are flexible enough to take into account that a given vantage point \(v\) could lie before or after a given retirement age \(T\).

We break this problem into an inner problem and an outer problem. The inner problem solves for the optimal consumption and savings for a fixed retirement date. The outer problem solves for the optimal retirement date. To compress notation, let \(y(t)\) be disposable income, which is normalized to 1 if \(t \in [0, T]\) and 0 otherwise.

Inner Problem

In the inner problem, we can ignore the third and fourth integrals in the objective functional, so that the inner problem is a standard two-stage control problem. Form a pair of Hamiltonians

\[
\mathcal{H}_1 = B(v - t)\frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda_1(t)[y(t) - c(t)], \text{ for } t \in [0, v], 
\]

(D2)

\[
\mathcal{H}_2 = F(t - v)\frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda_2(t)[y(t) - c(t)], \text{ for } t \in [v, T]. 
\]

(D3)

The first-order conditions include

\[
\frac{\partial \mathcal{H}_1}{\partial c(t)} = B(v - t)c(t)^{-\sigma} - \lambda_1(t) = 0, \text{ for } t \in [0, v], 
\]

(D4)

\[
\frac{\partial \mathcal{H}_2}{\partial c(t)} = F(t - v)c(t)^{-\sigma} - \lambda_2(t) = 0, \text{ for } t \in [v, T], 
\]

(D5)

\[
\frac{d\lambda_1(t)}{dt} = -\frac{\partial \mathcal{H}_1}{\partial k(t)} = 0, \text{ for } t \in [0, v], 
\]

(D6)
\[
\frac{d\lambda_2(t)}{dt} = -\frac{\partial H_2}{\partial k(t)} = 0, \text{ for } t \in [v, \bar{T}],
\]
\[
\lambda_1(v) = \lambda_2(v).
\]  

(Solving the costate equations together with the matching condition gives)

\[
\lambda_1(t) = \lambda_2(t) = \lambda, \text{ for } t \in [0, \bar{T}].
\]  

(Rewrite the Maximum Conditions)

\[
c(t) = \lambda^{-1/\sigma} B(v - t)^{1/\sigma}, \text{ for } t \in [0, v],
\]

\[
c(t) = \lambda^{-1/\sigma} F(t - v)^{1/\sigma}, \text{ for } t \in [v, \bar{T}].
\]  

(From the state equation we have)

\[
k(t) = \begin{cases} 
\int_0^t \{ y(s) - \lambda^{-1/\sigma} B(v - s)^{1/\sigma} \} \, ds, & \text{for } t \in [0, v], \\
 k(v) + \int_v^t \{ y(s) - \lambda^{-1/\sigma} F(s - v)^{1/\sigma} \} \, ds, & \text{for } t \in [v, \bar{T}].
\end{cases}
\]

(Evaluate \(k(t)\) at \(t = T\) and use \(k(T) = 0\))

\[
0 = \int_0^v \{ y(s) - \lambda^{-1/\sigma} B(v - s)^{1/\sigma} \} \, ds + \int_v^\bar{T} \{ y(s) - \lambda^{-1/\sigma} F(s - v)^{1/\sigma} \} \, ds
\]

and then solve for \(\lambda\)

\[
\lambda^{-1/\sigma} = \frac{\int_0^\bar{T} y(s) \, ds}{\int_0^v B(v - s)^{1/\sigma} \, ds + \int_v^\bar{T} F(s - v)^{1/\sigma} \, ds}.
\]

(Hence, looking backward over \(t \in [0, v]\) the optimal consumption path is)

\[
c^*_B(t|T, v) = \frac{\int_0^T y(s) \, ds}{\int_0^v B(v - s)^{1/\sigma} \, ds + \int_v^\bar{T} F(s - v)^{1/\sigma} \, ds} B(v - t)^{1/\sigma},
\]
and looking forward over $t \in [v, \bar{T}]$ the optimal consumption path is

$$c_F^*(t|T, v) = \frac{\int_0^T y(s)ds}{\int_0^v B(v-s)^{1/\sigma}ds + \int_v^\bar{T} F(s-v)^{1/\sigma}ds} F(t-v)^{1/\sigma}.$$  \hspace{1cm} (D16)

**Outer Problem**

Now that we have found the optimal consumption allocations for the past and future, conditional on a fixed retirement age $T$, the solution to the outer problem is given by:

$$T^*(v) = \arg \max_T \left[ \int_0^v B(v-t) c_B^*(t|T, v)^{1-\sigma} dt + \int_v^\bar{T} F(t-v) c_F^*(t|T, v)^{1-\sigma} dt \right. \left. - \int_0^{\min\{v,T\}} B(v-t) \psi dt - \int_v^{\max\{v,T\}} F(t-v) \psi dt \right].$$  \hspace{1cm} (D17)

Use the inner solutions $c_B^*(t|T, v)$ and $c_F^*(t|T, v)$, as well as $\lambda^{-1/\sigma}$ from the inner problem, to rewrite

$$T^*(v) = \arg \max_T \left[ \frac{T}{1-\sigma} \lambda - \int_0^{\min\{v,T\}} B(v-t) \psi dt - \int_v^{\max\{v,T\}} F(t-v) \psi dt \right].$$  \hspace{1cm} (D18)

After obtaining $T^*(v)$, we can define the solution consumption allocation

$$c^*(t|T^*(v), v) \equiv \begin{cases} 
  c_B^*(t|T^*(v), v), & \text{for } t \in [0, v], \\
  c_F^*(t|T^*(v), v), & \text{for } t \in [v, \bar{T}].
\end{cases}$$  \hspace{1cm} (D19)
References


Figure 1. Discount Functions from the Vantage Point of Retirement

The forward discount rate, $\rho_F$, is set to 0.02.
The forward discount rate, $\rho_F$, is set to 0.02.
The forward discount rate, $\rho_F$, is set to 0.02.
The forward discount rate, $\rho_F$, is set to 0.02. Each function is normalized to ensure a peak at unity.
Figure 5. Socially Optimal (Utilitarian) Consumption

The forward discount rate, $\rho_f$, is set to 0.02.
Figure 6. Socially Optimal (Utilitarian) Savings

The forward discount rate, $\rho_F$, is set to 0.02.
Figure 7. Optimal Retirement Age under Full Control Rights

The forward discount rate, $\rho_F$, is set to 4%.
The forward discount rate, $\rho_F$, is set to 4%, and the vantage point is $v = 15$ (age 40).
Figure 9. Savings Decomposition with $\rho_B = \rho_F$: The Role of Retirement

The forward discount rate, $\rho_F$, is set to 4%, and the vantage point is $v = 15$ (age 40).
The forward discount rate, $\rho_F$, is set to 4%, and the vantage point is $v = 15$ (age 40).
Figure 11. Savings Decomposition with $\rho_B=0$: The Role of Retirement

The forward discount rate, $\rho_F$, is set to 4%, and the vantage point is $v = 15$ (age 40).