

A Theory of Liquidity in Private Equity

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[Very Preliminary - Please Do Not Circulate]

Abstract

This paper studies the liquidity properties of private equity as an asset class. We first propose a model of delegated investment to endogenize some real-world institutional features of the sector. General partners (GPs) possess superior investment skills and raise capital from Limited Partners (LPs) to finance illiquid projects within funds. LPs who are not sensitive to liquidity risk, can earn a premium reflecting the scarcity of capital that can be committed on a long-term basis. We use the model to study the effect of a secondary market for private equity on liquidity premia and fundraising in the primary market. In particular, we show that partnership claims may trade at a discount in the secondary market when aggregate liquidity is scarce.

Keywords: Private Equity, Liquidity, Secondary Markets.

1 Introduction

Private equity firms are stewards of other people's capital. In practice, they operate through investment vehicles organized as limited partnerships, in which the general partners (GPs) - employees of the private equity firm itself - raise capital from outside investors, known as limited partners (LPs). These partnerships are typically structured as closed-end funds, in which the LP initially pledges (but does not provide) capital that is then provided over time as the GP identifies investment opportunities. The limited partnership agreements that bind the LP and GP in the partnership greatly constrain the ability of LPs to withdraw capital, and often associate severe penalties with renegeing on pledged commitments. In addition, private equity investments take time to mature, which delays the distribution of cash flows to investors. LPs typically need to wait several years before realizing a positive return on their investment (an effect known as the J-curve).

These features of private equity partnerships create a number of distinct concerns about liquidity in the eyes of many observers. There is a broad academic consensus that private equity investment has outperformed observationally similar (but more liquid) public equity investments over the last thirty years. However, it is not clear whether excess returns are sufficient to compensate LPs for the peculiar liquidity exposure that they face. This paper aims to answer this question, accounting for the recent developments of the private equity market. Over the last twenty years, a secondary market for partnership interests has emerged, allowing LPs to cash out on their investment before underlying assets mature. Anecdotal evidence also suggests that GPs are using early secondary buy-outs as a way to provide liquidity to investors.

Our main objective is to understand the effect of a secondary market for private equity on liquidity premia and fundraising in the primary market. To do so, we develop a model of private equity investment that captures several important institutional features of these partnerships. We first provide a rationale for LPs to tie up capital in illiquid fund structures based on an agency problem with GPs. We model LPs as investors who face liquidity shocks before private fund returns materialize. The possibility for early exit by LPs increases their appetite for private equity and reduces the cost of capital for LPs. We then analyze whether and how different secondary market arrangements may satisfy the liquidity needs of LPs.

In our model, GPs have specific skills to invest in illiquid projects. To leverage these skills, they must raise capital from investors, called LPs. The key friction of the model is an agency problem. GPs must exert effort for projects to succeed but effort is not observable by LPs. Performance-related fees incentivize effort by GPs who have skin in the game. The lower this fee, the larger the fund and the larger the total profits GPs

can earn. A standard result is that financing multiple projects allows GPs to reduce the fee they need to charge. Indeed, GPs can cross-pledge income across multiple projects and reduce the agency cost. In other words, the moral hazard friction rationalizes the whole-fund compensation structure for GPs, observed in these partnerships. Our model thus generates a realistic fund structure for delegated investments such as private equity. As expected, our model predicts that private equity investment increases when expected returns are high or LPs face little exposure to liquidity shocks.

The total profit attained by GPs depends on the clientele of private equity investors. LPs with little exposure to liquidity risk require a lower liquidity premium to commit capital on a long-term basis. This lowers the cost of financing for GPs who can raise larger funds. This pecking order for investors in private equity funds has implications when GPs face heterogeneous investors. When LPs with low liquidity exposure have little capital, they earn a premium beyond the fair compensation for liquidity risk. However, these returns may still be too low for LPs with high liquidity risk who stay out of the private equity market. Hence, the competitive pressure from these investors is limited by their lower capacity to bear illiquidity. However, as profitability of private equity increases, LPs with little liquidity exposure earn a higher premium and LPs who are very sensitive to liquidity risk enter the market.

In the second part of the paper, we use our model to study the effect of a secondary market for private equity. LPs who face liquidity shocks wish to sell their claim to new investors. This early exit option is beneficial to investors and ultimately reduces the cost of capital for GPs. However, secondary market liquidity and prices depend on the availability of capital to secondary buyers. In our model, this dry-powder for secondary market investment is endogenous since investors may initially decide to invest in the primary market or keep resources to participate in the secondary market. When aggregate liquidity shocks hit, many investors try to sell their participation in funds. However, few investors stand ready to buy these claims. Hence, when investors' exposure to liquidity shocks is high, secondaries trade at a discount and buyers earn an illiquidity premium. Hence, liquidity is scarce in the secondary market when it is most needed.

We compare the market for secondaries to secondary buy-outs as a source of liquidity for investors. Secondary buy-outs are sales of firms or projects by GPs to other buyers including other private equity funds. We argue that secondary buy-outs have one major drawback compared to secondaries since project sales can negatively affect market discipline. Indeed, GPs who can flip projects early have less incentives to work towards the success of this project. Hence, liquidity in the secondary market comes at an agency cost in the primary market. We thus show that the market for secondary buy-outs is primarily made on the sell side by funds with investors that are more ex-

posed to liquidity risk. These funds may also have a lower returns due to the negative effect of secondary market on incentives.

Relation to the literature

This paper analyzes the liquidity properties of private equity investment in a model where some key institutional features are endogenized. In this respect, our rationale for LPs to finance several projects through a fund structure is similar to [Axelson et al. \(2009\)](#). Like in this paper, investing through funds rather than deal-by-deal creates some “inside equity” for the GP, which alleviates moral hazard. In their case, the quality of projects is the GP private information. Cross-pledging or projects’ cash flows prevents the GP from investing in some bad projects, if they have the hope of finding good projects in the future. [Axelson et al. \(2009\)](#) also analyze the role of third-party debt financing to avoid over-investment in bad projects. However, they do not consider the illiquidity or private equity investment or the role of the secondary market.

Other papers study the persistence of private equity returns over time. [Hochberg et al. \(2014\)](#) provide a model where existing LPs learn the GPs’ skill over time, giving rise to informational holdup in a setting with overlapping funds. They show that this informational holdup reduces the ability of good GPs to increase fees in their next time, leading to performance persistence across funds, consistent with the empirical evidence in [Kaplan and Schoar \(2005\)](#).¹ In this paper, GPs do not compete on fees but rather on size when facing different investors.

While they do not derive an optimal fund structure, [Lerner and Schoar \(2004\)](#) argue that GPs endogenously limit trading of Limited Partnership claims to screen for “good” LPs. Similar to our model, LPs can be hit by liquidity shocks and then wish to exit their investment through a sale of the partnership claim. In their model, LPs also acquire private information about the skills of GPs during the investment. Outside investors only see whether incumbent LPs sell their claim but do not know why. Hence, observing a sale is a bad signal for future LPs and raises the cost of capital for the GP’s follow-up fund. Preventing LPs from selling their share may thus facilitate subsequent financing for GPs. Our contribution is to analyze the illiquidity premium in the secondary market when it is active.

[Sorensen et al. \(2014\)](#) also investigate the illiquidity cost of private equity to investors – i.e. the required excess return for investing in private rather than public equity – in a dynamic portfolio choice model. In their paper, the cost of private equity is that it exposes a risk-averse LP to additional uninsurable risk. In our paper, in con-

¹As [Harris et al. \(2014\)](#) show, the performance persistence in VC is much stronger than in buyouts. Also, [Korteweg and Sorensen \(2017\)](#) argue that this performance persistence is not investable, since the ultimate performance of the previous fund is not known at the time the GP raises the next fund.

trast, LPs are risk neutral, but suffer from liquidity shocks, as in [Diamond and Dybvig \(1983\)](#). Our paper and thus our measure of illiquidity accounts for the presence of a secondary market. Using a proprietary data set of secondary markets bid, [Albuquerque et al. \(2017\)](#) find evidence that buyers earn higher expected return in response to liquidity shocks, in line the predictions of our model.

2 Model

The economy lasts for three periods, denoted $t = 0, 1, 2$. The economy is populated with investors or LPs who have resources and managers or GPs who can manage projects but have limited resources. GPs seek financing from LPs to leverage their investment skills. LPs face liquidity shocks which reduces their willingness to commit capital for long-term investments.

LPs

LPs are risk-neutral and consume in period 1 and 2. They may experience a liquidity shock in period 1 that increases their discount rate. We model the preferences of LPs in the spirit of Diamond-Dybvig consumers. A LP with liquidity risk λ then has the following preferences.

$$u(c_1, c_2) = c_1 + \tilde{\delta}c_2, \quad \text{where } \tilde{\delta} = \begin{cases} 1 & (1 - \lambda) \\ 0 & \lambda \end{cases} \quad (1)$$

where c_t is period t consumption. A LP with a high λ favors liquid investments which pay off in period 1. LPs initially own cash they can store at a risk-free rate of $r = 0$.² There is a large continuum of mass M of LPs with one unit of cash each at date 0. We will first assume that LPs are ex-ante identical, that is they all face the same probability of a liquidity shock λ . We relax this Assumption in Section 4. We assume that M is large so that GPs capture the gains from trade. Project returns are independent across dates.

GPs and Projects

There is a mass 1 of managers or GPs who are risk-neutral and do not discount future cash flows. GPs have an initial endowment of 1 unit of cash at date 0.³ GPs can

²If $r > 0$, LPs would have a preference for late capital calls that can be financed out of the returns from this outside option.

³Total investment in private equity will be a multiple of GPs' resources. Hence, the model would naturally predict that investment in private equity rises after good times when GPs have been succesful.

run scalable projects at date 0 and at date 1. All projects mature at date 2 and pay either R in case of success or 0 per unit of investment. We assume that a GP can only start one project per date but that he can manage two projects started at different dates.

Moral Hazard and Project Return

Projects have positive net present value only if the GPs exert effort at the investment date. We follow the approach of [Holmstrom and Tirole \(1997\)](#) and assume that effort is not observable by LPs. When GPs exert effort, a project succeeds with probability p . When GPs shirk, the probability of success goes down from p to q but GPs obtain a private benefit B per unit of funds invested in the project. Our assumption that exerting effort (resp. shirking) is efficient (resp. inefficient) writes

$$pR \geq 1 \geq qR + B \tag{2}$$

where the rightmost term is the total monetary and non-monetary payoff from a project when GPs shirk. The moral hazard problem implies that not all the cash flows from a project are pledgeable to outside investors.

Limited pledgeability can constraint project financing if the income pledgeable to outside investors does not cover their initial investment. In our setup, this condition writes

$$p \left(R - \frac{pB}{p^2 - q^2} \right) < 1 \tag{3}$$

The analysis will show that the left hand side is the maximum return the GPs can pledge per unit of investment in a fund. The reader may be more familiar with the condition

$$p \left(R - \frac{B}{p - q} \right) < 1 \tag{3b}$$

which prevents self-financing of a project in the Holmstrom-Tirole model. Condition (3) is stronger than (3b) because, as we will show, joint financing of two projects is easier than financing of a single project. When (3) holds, LPs do not invest with GPs unless the latter have *skin in the game* through some investment in their own funds.

Contracts

GPs offer contracts for investment partnerships with LPs at date 0. The contract specifies the share of the total investment allocated to the date 0 and the date 1 projects, denoted x and $1 - x$ respectively, and the compensation schedule of the GPs. The total fund size is denoted by I and will be determined in equilibrium.

Secondary Market

In Section 5, we introduce a secondary market for private equity at date 1. We will first consider the possibility for LPs to sell their claims in the fund, that is a market for secondaries. Then, we will allow GPs to sell projects in a market for secondary buy-outs.

3 Fund Design

In this section, we derive the optimal fund design without a secondary market. We will first characterize the optimal split of the total fund size into the two projects. Then, we will derive the minimum compensation schedule for the GP to exert effort on both projects. Finally, we will characterize the fund size as a function of the liquidity risk faced by LPs, which determines the cost of capital for GPs.

It will be useful to introduce some notation. Observe that the total fund payoff is given by yI where

$$y \in \{0, R(1-x), Rx, R\}$$

The extreme payoffs correspond to a joint failure or a joint success while the middle payoffs correspond to a single success. Given, the linearity of the problem, the compensation of the GP can be written $Iw(y)$ where $w(y)$ is the compensation received by the GP per unit of investment when the unit payoff from the fund is y . We impose the standard monotonicity constraints on the manager compensation that

$$y_2 \geq y_1 \quad \Rightarrow \quad 0 \leq w(y_2) - w(y_1) \leq y_2 - y_1 \quad (4)$$

Condition (4) states that the GP compensation must increase with the fund payoff but that it cannot increase more than one for one with that payoff. The first part of the condition is stated for completeness although it arises as an equilibrium property in a model with moral hazard. The second part of the compensation is usually motivated to avoid cash flow manipulation by the manager.⁴ Figure 1 summarizes the timeline of the model.

Capital calls

The share of capital $x \in [0, 1]$ allocated to the first project is part of the contract design. Observe that all agents are risk-neutral, that projects started at date 0 or date 1 are otherwise identical and the moral hazard problem is separable across projects.

⁴ Suppose that the manager compensation schedule instead verifies $y_1 - w(y_1) > y_2 - w(y_2)$. Then, when the payoff is y_1 , the manager would claim that it is in fact y_2 , borrow $y_2 - y_1$ at the market rate of 0 and leave $y_2 - w(y_2)$ to the investors.

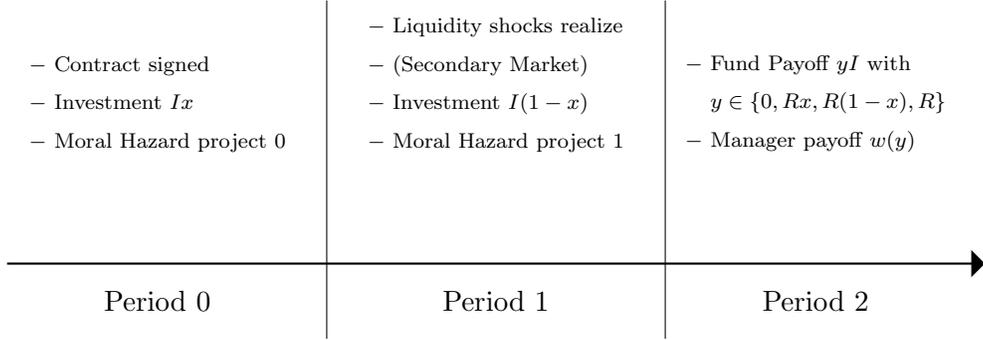


Figure 1: Timeline

One may thus think that the choice of x is indeterminate. However, it is always weakly optimal for GPs to diversify across projects to reduce the severity of the moral hazard problem. Bundling investments in a fund allows GPs to cross-pledge the income of two projects. The bonus paid for the success of a project can be recycled to incentivize effort on the second project.⁵ In fact, due to the symmetry of the problem, an equal split is optimal.

Lemma 1 (Diversification Benefits)

It is optimal to split investment equally across projects, that is $x = \frac{1}{2}$.

We prove this result in the Appendix showing that GPs cannot increase profit over the level in Proposition 1 using a different split across projects. The proof shows that there exists in fact a range $[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ where ϵ is small and depends on the parameters of the model. This indeterminacy follows from the perfect symmetry projects. If the second project with shorter maturity had a lower return than the first project, GPs would tend to call more capital early. However, unless the difference in expected return is very large, there are still benefits from diversification across projects.

To get some intuition about the result, let us normalize the total investment to 1 and compare a fund where $x = 0$ to a fund where $x = 1/2$. We argue that the GP can charge a lower fee in the fund with split investments. As we will show, this is beneficial because the GP can then raise a larger fund. When $x = 0$, all investment is made in period 0. The fund payoff is either 0 or R . The minimum fee $\underline{w}(R)$ the GP must charge

⁵This insight is similar to the benefit of diversification in the context of delegated monitoring, originally due to Diamond (1984). Axelson et al. (2009) rely on a similar argument to model private equity funds.

in case of success to exert effort is pinned down by his incentive constraint. We have

$$p\underline{w}(R) = q\underline{w}(R) + B \quad \Rightarrow \quad \underline{w}(R) := \frac{B}{p - q}$$

The expected compensation is then given by $p\underline{w}(R)$.

Consider now the fund with split investment such that $x = 1/2$. The possible payoffs are 0, $R/2$ and R . It is of course possible to compensate the GP for each project independently of the outcome of the other project. By linearity and given that projects returns are independent, the expected compensation of the GP will be again equal to $p\underline{w}(R)$. This means that the cost of providing incentives would be the same than in a fund with $x = 0$. However, this compensation scheme is sub-optimal in the fund with $x = \frac{1}{2}$.

Suppose instead that the GP receives a bonus of $\frac{B}{p(p-q)}$, paid only when both projects succeed. Note that the the expected compensation is still equal to $p\underline{w}(R)$ since the bonus is paid with probability p^2 . We are left to show that the GP strictly prefers to exert effort than to shirk. It is easy to verify that the best deviation for the GP is to shirk on both projects. With a bonus of $\frac{B}{p(p-q)}$ paid for a joint success, the GP strictly prefers to work since

$$\frac{pB}{(p - q)} > \frac{q^2 B}{p(p - q)} + B$$

where the inequality holds because $p > q$. The left hand side is the expected compensation of the GP when he exerts effort. The right hand side is the expected payoff is he shirks on both projects. Since the incentive constraint is slack, it is possible to reduce the bonus paid to the GPs for a joint success. This lowers the expected cost of effort and thus the return pledgeable to the LPs, which in turn allows GPs to raise more funds. The following paragraphs formalize this intuition to derive the minimum compensation that is incentive compatible.

Minimum compensation

Given the result in Lemma 1, the fund payoff is either 0, $R/2$ or R if both projects fail, one project succeeds and both projects succeed, respectively. We can rewrite the monotonicity constraints in equation (4) explicitly as

$$0 \leq w(R) - w(R/2) \leq R/2 \tag{5}$$

$$0 \leq w(R/2) \leq R/2 \tag{6}$$

The minimum compensation for the GP is the compensation schedule that induces effort at a lower cost. Observe that the GP has two potential deviations. He can shirk on one

project only or on both projects. We guess and verify that the relevant deviation for a GP is to shirk on both projects. Hence at date 0, a GP who exerts effort (resp. shirks) anticipates that he will exert effort (resp shirk) at date 1. The incentive constraint at date 0 thus writes

$$p^2w(R) + 2p(1 - p)w(R/2) \geq q^2w(R) + 2q(1 - q)w(R/2) + B \quad (\text{IC})$$

When a GP exerts effort, the probability of a joint success is p^2 while the probability of a single success is $2p(1 - p)$ since this state is reached either when the first or the second project fails. The expression when shirking on the right hand side is similar, replacing the success probability by q and accounting for the private benefit B .

The minimum compensation solves the following problem

$$\begin{aligned} \min_{\{w(R/2), w(R)\}} W &= p^2w(R) + 2p(1 - p)w(R/2) & (7) \\ \text{subject to} & \text{(IC)} \end{aligned}$$

For each payoff $y \in \{R/2, R\}$, we can define an efficiency ratio $\alpha(y)$ equal to the marginal change in the incentive constraint (IC) divided by the marginal change in the expected compensation W from an increase $w(y)$. This ratio α measures the incentives gain from a state-contingent bonus normalized by the probability that this bonus is paid. We have

$$\alpha(R) = \frac{p^2 - q^2}{p^2} > \frac{2p(1 - p) - 2q(1 - q)}{2p(1 - p)} = \alpha(R/2)$$

This means that it is strictly more efficient to compensate the GP after a joint success than after a single success. Intuitively, a joint success is harder to achieve when shirking and is thus a good signal of effort. Since the GP is risk-neutral, he should be compensated only after a joint success. Observe however that the monotonicity constraint (5) imposes that $w(R) \leq R/2$. This means that the GP is effectively compensated for two projects out of the cash flows from one project only. When R is too low, this might not be feasible. Hence, as the following Lemma shows, the GP's payoff is less convex in the fund payoff when R is low.

Lemma 2 (Fund Compensation)

The optimal compensation in a fund is given by

$$(w^*(R/2), w^*(R)) = \begin{cases} \left(\frac{1}{2-p-q} \left[\frac{B}{p-q} - (p+q)\frac{R}{2} \right], w(R/2) + R/2 \right) & \text{if } R \leq \frac{2B}{p^2-q^2} \\ \left(0, \frac{B}{p^2-q^2} \right) & \text{otherwise} \end{cases} \quad (8)$$

As we observed, it is optimal to compensate the GP only for a joint success. Setting $w(R/2) = 0$ and having the incentive constraint (IC) bind gives the expression for $w(R)$ when $R \geq \frac{2B}{p^2 - q^2}$. As we explained, the monotonicity constraint (5) binds for low values of R . This means that GPs must also be compensated after a single success. In other words, the compensation schedule becomes less convex in the fund payoff.

We verify our initial guess that a single deviation to shirk only on one project is not profitable. Focusing on the case where R is high, the condition writes

$$p^2 w^*(R) \geq qp w^*(R) + \frac{B}{2} \quad \Leftrightarrow \quad w^*(R) \geq \frac{B}{2p(p-q)}$$

It is easy to verify that the inequality holds for the minimum compensation since $2p > p + q$. Similar computation yield the result for the other cases.

Lemma 2 also confirms that it is more efficient to run two projects within a fund rather than financing them independently. To see this, let us compute the minimum expected compensation of the GP

$$W^* = \begin{cases} \frac{pB}{p-q} - \frac{pq}{2-p-q} \left[R - \frac{B}{p-q} \right] & \text{if } R < \frac{2B}{p^2 - q^2} \\ \frac{pB}{p-q} - \frac{q}{p+q} \frac{pB}{p-q} & \text{otherwise} \end{cases} \quad (9)$$

Observe that in each case, W^* is lower than the expected compensation of the GP when running a single project, which is equal to $pB/(p-q)$. When R is large, the relative gains are larger because the benefits from diversification can be fully captured. In the following, we show that the GP's ability to charge lower fees when combining projects in funds allows them to increase fund size and make more profit.

Fund Size

In the market for funds, the surplus is captured by GPs because LPs' resources M are assumed to be large. GPs propose contracts specifying a compensation schedule $w()$. It is in fact possible to characterize a compensation schedule by the expected rent or fee W . Observe first that no GP will offer a contract with $W < W^*$ since LPs anticipate that the GP will then shirk. What is then the optimal compensation schedule? Observe that the expected fee has two effects on the total profit of the GP. On the one hand, a higher expected fee raises the GP's profit for a given fund size. However, the expected fee also reduces the fund size I . Indeed, I is pinned down by the participation constraint of LPs

$$(1 - \lambda) [pR - W] I = I - 1 \quad (10)$$

The right hand side $I - 1$ is the LPs' contribution to the fund. LPs make zero profit when the expected payoff from the fund multiplied by the liquidity discount $(1 - \lambda)$

just compensates for this initial contribution. Observe that the expected fee W lowers the expected return for LPs (also called the pledgeable amount). It thus decreases the fund size I . The optimal compensation schedule for the GP ultimately solves

$$\max \Pi_{GP} = WI = \frac{W}{1 - (1 - \lambda)[pR - W]} \quad \text{subject to} \quad W \geq W^* \quad (11)$$

Since both the numerator and the denominator are linear in W , the solution is either W^* or 0, that is either GPs charge the minimum expected fee to maximize the fund size or they do not seek outside financing from LPs. The answer depends on the return required by LPs to be compensated for the fund illiquidity.

Proposition 1 (Fund Clientele)

GPs raise capital from LPs to invest in illiquid assets within a fund if

$$\lambda \leq \bar{\lambda}(p, R) := \frac{pR - 1}{pR} \quad (12)$$

The fund total capital is split equally across projects and the optimal compensation schedule is given in Lemma 2.

Proposition 1 follows directly from the optimization program (11). When $\lambda > \bar{\lambda}(p, R)$, the fund's assets are too illiquid for the LPs since then

$$(1 - \lambda)pR < 1$$

This condition has a very simple interpretation: LPs would not invest in private equity even if there could manage projects themselves. In this case, GPs cannot raise financing from LPs even if they charge the minimum fee. The function $\bar{\lambda}(p, R)$ thus defines a frontier in the cross-section of investors above which investors stay out of the private equity market because of illiquidity. It follows directly from the expression of $\bar{\lambda}(p, R)$ in equation (12) that LPs with lower tolerance to illiquidity will invest in private equity as expected returns pR go up. In Section 5, we study how the presence of a secondary market affects this frontier.

Proposition 1 allows us to derive the fund size and the GP profit as a function of the parameters of the model. Focusing on the case where $R \geq \frac{2B}{p^2 - q^2}$, we have

$$I^*(\lambda) = \frac{1}{1 - (1 - \lambda)p \left[R - \frac{pB}{p^2 - q^2} \right]} \quad (13)$$

$$\Pi_{GP}^*(\lambda) = \frac{p^2 B}{p^2 - q^2} I_{GP}^*(\lambda) \quad (14)$$

It is immediate that an increase in the payoff R increases the fund size and the GPs' profit. As private equity becomes more profitable, GPs can attract more capital and raise larger funds. This increases their profit since the expected compensation is independent of R . The profitability of projects can also be captured with the probability of success p and similar results obtain. Finally, equations (13) and (14) show that fund size and GPs' profit are decreasing in λ . LPs with lower liquidity risk provide cheaper financing to GPs since they do not require a high liquidity premium to invest in private equity. GPs who face a lower cost of capital can then raise larger funds and increase their profit. The following corollary confirms that these results hold independently of the parameter configuration.

Corollary 1 (Comparative Statics)

Fund size I^ and GPs' profit Π_{GP}^* are decreasing in LPs' liquidity risk λ , increasing in the projects' payoff R and increasing in the probability of success q when $p = q + \alpha$ and α is a fixed parameter.*

In the comparative statics for the probability of success, it is useful to fix the difference $p - q$. Having both probabilities increase together allows us to focus on the impact of a general increase in profitability. Increasing the probability of success p conditional on effort but not q would also affect the marginal return from effort. The proof shows that the effect of an increase in R on the GPs' profit is even stronger when $R < \frac{2B}{p^2 - q^2}$ since then projects become more profitable and the monotonicity constraint (5) is relaxed.

We conclude this section by summarizing the main results. First, the model provides a rationale based on moral hazard to explain why GPs will bundle several investments into a fund, like in private equity. Bundling investments allows LPs to tie the compensation of the GPs to the joint outcome of the projects which is typical of the compensation received by GPs in private equity. When LPs face illiquidity risk, the model also shows that LPs that are less sensitive to this risk are better investors for GPs since they require a lower liquidity premium. This decreases the cost of capital for GPs who can raise larger funds.

4 Heterogeneous LPs

In our analysis so far, GPs were facing a large homogeneous population of LPs with the same sensitivity to liquidity shocks λ . In practice, there is a large potential range of investors who may consider investing in private equity. Still, investors willing to commit capital on a long term basis may scarce. To capture this heterogeneity, we allow for

two types of LPs characterized by their probability to receive a liquidity shocks λ_L and λ_H where

$$\lambda_L < \lambda_H < \bar{\lambda}(p, R) \quad (15)$$

The left inequality ensures that λ_L -LPs are less sensitive to liquidity shocks. The second inequality ensures that even λ_H -LPs are suitable investors for private equity, according to Proposition 1. Each type of LP has a mass of m_L and $m_H = M - m_L$ where M represents all the resources available in the economy.

The goal of this section is to determine the fund industry equilibrium when GPs face heterogeneous LPs. GPs offer contracts to LPs specifying the allocation of total investment x and the compensation schedule $w(\cdot)$. LPs supply capital to GPs as long as their net expected return from investing in private equity exceeds the short-term outside option with return 0.

Before stating the Proposition, we first discuss some intuitive properties of the equilibrium. Proposition 1 showed that the same fund structure is optimal for any type of LP who can invest in private equity. Hence, GPs will offer the same contract to all LPs in equilibrium. By symmetry, all funds will have the same size I since a GP with a larger fund would make strictly more profit. Taking fund size I as given, let us compute the net expected return $r_{LP}(\lambda)$ from investing in private equity for a LP with liquidity risk $\lambda \in \{\lambda_L, \lambda_H\}$. Using the participation constraint, equation (10), we have

$$r_{LP}(\lambda) = \frac{(1 - \lambda)(pR - W)I^*}{I^* - 1} - 1$$

The term at the numerator is the total distribution to LPs weighted by the liquidity discount $(1 - \lambda)$ while the denominator is the total contribution of LPs to the fund. Suppose then that λ_H -LPs are active in equilibrium which requires that $r_{LP}(\lambda_H) \geq 0$. This immediately implies that λ_L -LPs net expected return is strictly positive since $r_{LP}(\lambda_L) > r_{LP}(\lambda_H)$, that is capital supplied by λ_L -LPs earns a premium. We are left to determine the equilibrium fund size I^* to quantify the premium earned by λ_L -LPs.

Proposition 2 (Heterogeneous LPs)

Suppose that $\lambda_L < \lambda_H < \bar{\lambda}(p, R)$ and $M > I^*(\lambda_L)$. The equilibrium fund size I^* is weakly-increasing in the share of capital in the hands of λ_L -LPs. We have

$$I^* = \max \left\{ \min\{m_L, I^*(\lambda_L)\}, I^*(\lambda_H) \right\} \quad (16)$$

The next expected return earned by λ_L -LPs reflects the scarcity of their capital m_L

$$r_{LP}^*(\lambda_L) = \frac{I^*(\lambda_L) - I^*}{I^*(\lambda_L)(I^* - 1)} \quad (17)$$

Other LPs break even, that is $r_{LP}(\lambda_H) = 0$.

Proposition 1 showed that GPs offer funds with the same compensation structure and the same pattern of capital calls to all types of LPs. However, GPs prefer to raise capital from LPs with lower liquidity risk λ_L . Indeed, these investors require a lower compensation for illiquidity. This reduces the cost of capital for GPs who can then raise larger funds and increase total profit. Hence, the key variable shaping the equilibrium in Proposition 2 is the capital m_L of the λ_L -LPs rather than the total resources M available for private equity.

When m_L is large, “long-term capital” is abundant and the equilibrium is as if GPs were facing an homogeneous population of λ_L -LPs. Competitive pressure prices other LPs out of the market since expected returns are too low to compensate for their illiquidity risk. When m_L decreases, GPs cannot operate at full capacity $I^*(\lambda_L)$ and fund size is pinned down by the resources m_L available to λ_L -LPs. When m_L decreases further, λ_H -LPs enter the market as increasing expected returns in small funds eventually provide a fair compensation for their liquidity risk.⁶ Entry by other LPs limit the return $r_{LP}^*(\lambda_L)$ that λ_L -LPs can earn from investing in private equity. Equation (17) shows that this return depends on the scarcity of capital owned by λ_L -LPs and on the difference $\lambda_H - \lambda_L$, that is how special λ_L -LPs are compared to other investors. Figure 2 represents the main equilibrium outcomes as a function of the capital m_L available to λ_L -LPs.

Finally, we can study the effect of projects profitability on the private equity composition. Corollary 1 showed that the thresholds $I^*(\lambda_L)$ and $I^*(\lambda_H)$ increase as q or R goes up. This immediately implies that λ_L -LPs earn higher returns when expected

⁶Remember that the smaller the fund, the greater the co-investment by GPs and the greater the returns for LPs since the GPs’ compensation per unit of investment is fixed.

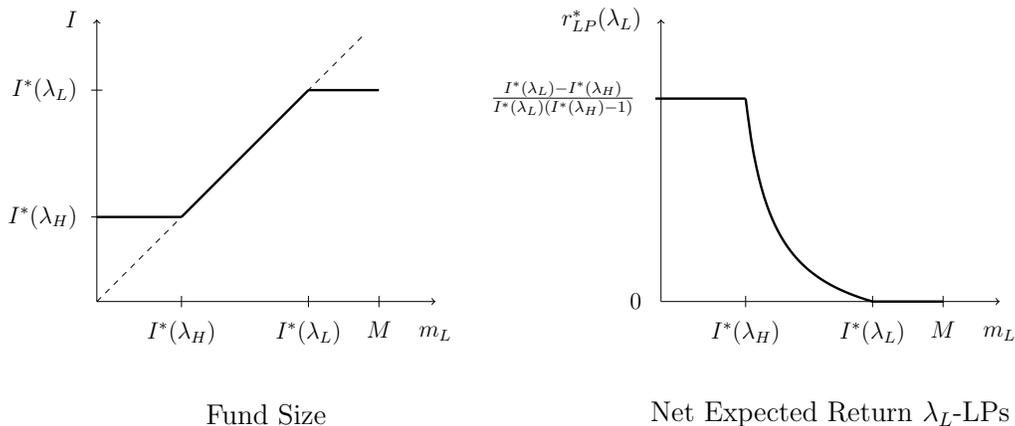


Figure 2: Equilibrium with heterogeneous LPs
 $\frac{m_L}{M}$ is the share of wealth of λ_L -LPs

returns are high but also that λ_H -LPs are more likely to invest in private equity.

5 Secondary Market

We now introduce a secondary market for private equity. LPs who commit capital for long-term investments may value an early exit from the fund when they face liquidity needs. We will consider two alternative liquidity sources for LPs. First, we will let investors trade claims in funds in a market for “secondaries”. Then, we will study secondary buy-outs as a way for GPs to provide liquidity to their investors.

5.1 Secondaries

At date 1, a market for secondary participation in the PE funds opens. A secondary participation is a claim on the existing assets together with an obligation to finance the remaining capital calls. We first model the market for secondaries as a friction-less competitive market. Our main result is that secondaries trade at a discount if buyers do not have enough dry-powder. In our model, these resources are endogenous since investors face a trade-off between investing in primary funds as LPs or staying put to buy participation from distressed LPs.

We assume that buyers only acquire the assets in the fund, that is the claim is net of any capital call. This means that LPs who wish to sell still need to finance

the capital call at date 1. This assumption comes without loss of generality although sellers precisely want to avoid the capital call. If buyers had to finance the capital call themselves, the price of the claim would indeed adjust downward to reflect the associated liability.⁷

The price of a secondary claim in the fund, normalized for the value of the LPs contribution, is denoted P_f . We first derive the equilibrium value of P_f at date 1. The fundamental value of the claim is given by

$$\bar{P}_f = \frac{(pR - W)I}{I - 1} \quad (18)$$

To understand this expression, observe that $(pR - W)I$ is the total distribution to all LPs. Since LPs contribute for $I - 1$ in the fund, the second term of (18) is the unit value of the assets to the LPs. The value of the claim to a distressed LP is simply $\underline{P}_f = 0$ since he does not enjoy any utility from the assets' payoff at date 2. Since $\underline{P}_f < \bar{P}_f$, there are gains from trade between distressed LPs and non-distressed investors.

To determine the equilibrium price P_f for secondaries, let us derive the supply and demand for secondaries at date 1. Distressed LPs supply all their claims if the price is above 0, that is

$$S = \lambda(I - 1), \quad P_f \geq 0 \quad (19)$$

Non-distressed investors use all their resources to buy claims if the price does not exceed \bar{P}_f . The total resources available to these investors is $(1 - \lambda)(M - (I - 1))$ where $(1 - \lambda)M$ is their initial resources and $(1 - \lambda)(I - 1)$ their own commitment to private equity in the primary market. Hence, we can write the demand for secondary claims as

$$D(P_f) = \begin{cases} \frac{(1 - \lambda)(M - (I - 1))}{P_f} & \text{if } P_f < \bar{P}_f \\ D \in \left[0, \frac{(1 - \lambda)(M - (I - 1))}{P_f}\right] & \text{if } P_f = \bar{P}_f \end{cases} \quad (20)$$

Two cases are possible. If supply is small compared to demand, the claim will trade at fundamental value \bar{P}_f . Using equations (19) and (20), this case arises if

$$(1 - \lambda)(M - (I - 1)) \geq \lambda(pR - W)I \quad (21)$$

Equation (21) has a natural interpretation. Secondaries trade at fundamental value if the fair value of the funds' assets in the hands of distressed LPs does not exceed the

⁷To be accurate, the equivalence breaks down if the price of the claim including the liability is restricted to be positive, that is if sellers cannot subsidize buyers to take over their commitment. However, this assumption would be ad hoc in our model.

resources available to non-distressed investors in the secondary market. If condition (21) does not hold, secondaries trade at a discount. The price P_f is determined by equating supply and demand. Using equations (19) and (20), we obtain

$$P_f = \frac{(1 - \lambda)(M - (I - 1))}{\lambda(I - 1)} \quad (22)$$

There is the “cash-in-the-market pricing” because the claim price is determined by the amount of resources available to buyers rather than by the fundamental value to these buyers. Observe that condition (21) and thus whether secondaries trade at a discount depend on fund size I and thus on the outcome of the primary market.

The presence of a secondary market affects the term of trades in the primary market. First, observe that the outside option for investors improves when secondaries trade at a discount. Indeed, a buyer in the secondary market earn a strictly positive return when $P_f < \bar{P}_f$. Hence, the ex-ante gross return on one unit of cash at date 0 is

$$1 + r(P_f) = \lambda + (1 - \lambda) \frac{\bar{P}_f}{P_f} \quad (23)$$

since investors may buy secondaries only when they do not receive a liquidity shock themselves. GPs who raise funds from LPs in the primary market must offer a rate of return at least equal to $r(P_f)$. The fund size is then determined by the participation constraint of LPs who must be indifferent between investing in the primary market and keeping cash to invest in the secondary-market. This condition writes

$$(1 - \lambda)(pR - W)I + \lambda(I - 1)P_f = (1 + r(P_f))(I - 1) \quad (24)$$

It is useful to compare the participation constraint (24) to its counterpart without a secondary market, equation (10). First, the liquidity of the fund increases since LPs can sell their participation. Hence, LPs can cash in early on their long-term investment. This increases their appetite for private equity and reduces the cost of capital for GPs. However, LPs only invest in private equity if the expected return exceeds the return on holding cash, equal to $1 + r(P_f)$.

Hence, fund size in the primary market is determined in a subtle way by the liquidity of the secondary market. As we have shown, secondary market liquidity depends in turn on the size of funds raised in the primary market. The following proposition describes the equilibrium with a secondary market.

Proposition 3 (Equilibrium with secondaries)

There exists two thresholds $0 < \underline{\lambda}_{SM} < \bar{\lambda}_{SM} < 1$ given by

$$\underline{\lambda}_{SM} = 1 - \frac{(pR - W)}{M[1 - (pR - W)]}, \quad \bar{\lambda}_{SM} = 1 - \frac{(pR - W)}{MW} \quad (25)$$

such that

- i) When $\lambda \leq \underline{\lambda}_{SM}$, investment is maximal $I_{SM}^*(\lambda) = I^*(0)$ and secondaries trade at par.
- ii) When $\lambda \in [\underline{\lambda}_{SM}, \bar{\lambda}_{SM}]$, investment is given by $I_{SM}^*(\lambda) = 1 + (1 - \lambda)M < I^*(0)$ and secondaries trade at a discount equal to

$$\frac{(1 - \lambda)M}{(pR - W)(1 + (1 - \lambda)M)}$$

- iii) When $\lambda \geq \bar{\lambda}_{SM}$, GPs do not raise capital from investors.

There are two main results in Proposition 3. First, a market for secondaries provides liquidity to distressed LPs who can cash in their investment before the underlying assets mature. These gains appear most clearly when investors face little exposure to liquidity shocks (low λ). Then, secondaries trade at par and LPs can realize the full value of their investment although they exit early. This reduces the cost of capital for GPs in the primary market who can raise bigger funds. When $\lambda \leq \underline{\lambda}_{SM}$, GPs realize the same profit that they would obtain if LPs were not exposed to liquidity shocks.

The second result however, is that liquidity provision on secondary market is limited when it most needed. Indeed, when investors face significant exposure to liquidity shocks (high λ), few buyers may purchase claims in the secondary market since most investors value liquidity. Simultaneously, when λ is high, many LPs seek to exit through the sale of secondaries. Hence, buyers provide liquidity only if they can earn a superior return and the secondaries trade at a discount. The size of the discount increases with the severity of the liquidity shock.

When $\lambda \geq \bar{\lambda}_{SM}$, secondaries provide limited liquidity to investors who are heavily exposed to liquidity shocks. Hence, despite the existence of a secondary market, private equity investments are too illiquid for investors who require a very high rate of return from GPs. In this case, GPs choose to invest without raising funds from LPs.

The main message from Proposition 3 is that liquidity does not come for free. Ob-

serve that we modeled secondary trading as a competitive process, thereby abstracting from many potential frictions affecting these transactions.⁸ However, in general equilibrium, the potential buyers in the secondary market tomorrow are today's investors. When the economy is hit by market-wide liquidity shocks, all investors value liquidity and few buyers can take over the private equity commitments. This implies that secondaries trade at a discount below net asset value. Eventually, liquidity provision in the secondary market is not sufficient for LPs to commit capital in the primary market for private equity. To summarize, liquidity is scarce when it is most needed.

5.2 Secondary Buy-outs

[To be added]

6 Conclusion

This paper provides a model of delegated fund investment to study the liquidity properties of private equity. Our model replicates several contractual features of private equity funds. Investors commit capital for a series of long-term investments and GPs receive compensation based on the overall performance of the fund. Investors exposed to liquidity shocks are reluctant to invest in private equity because assets are illiquid. This creates an equilibrium premium for capital provided by LPs with limited liquidity needs. We study the role of a secondary market in reducing the fundamental illiquidity of private equity funds. Our model rationalizes trade in secondaries at a discount below Net Asset Value. When the economy is exposed to illiquidity shocks, few buyers can make the secondary market while many LPs seek to exit their investment. When liquidity shocks are high, the extreme illiquidity of the secondary market leads to a funding dry-up in the primary market.

⁸For instance, there might be search and bargaining frictions or information frictions due to the difficulty to locate a buyer or the uncertainty about the fund's assets valuation.

Appendix

A Proofs

A.1 Proof of Lemma 1 and 2

To simplify the exposition, we introduce slightly different notation than in the main text. Let us denote by $w_F^{SS}(x)$ the GPs compensation at date 2 when both projects succeed. We let $w_F^{FS}(x)$ (resp. $w_F^{SF}(x)$) be the GPs compensation when the first (resp. the second) project fails but the other project succeeds. The GP's expected compensation is thus given by

$$W = p^2 w^{SS}(x) + p(1-p) [w^{FS}(x) + w^{SF}(x)] \quad (26)$$

Our objective is to derive the compensation schedule that minimizes the expected compensation W for a given value of x . Then, we want to show that the minimum over values of x is reached for $x = \frac{1}{2}$. By symmetry, it is enough to consider only the case $x \geq \frac{1}{2}$. In this case, the monotonicity constraints (4) can be written

$$0 \leq w^{SS}(x) - w^{SF}(x) \leq R(1-x) \quad (27)$$

$$0 \leq w^{SF}(x) - w^{FS}(x) \leq R(2x-1) \quad (28)$$

$$0 \leq w^{FS}(x) \leq R(1-x) \quad (29)$$

The proof is in several steps. We first show that GPs do not receive a positive compensation when one project fails unless the right inequality of the monotonicity constraint (27) binds. We then show that the expected compensation of the GP is minimized for $x = \frac{1}{2}$ for the different cases for R described in Lemma 2.

A.1.1 Optimality of joint compensation

We will show that unless the right inequality of constraint (27) binds, we can set $w_F^{FS}(x) = w^{SF}(x) = 0$ without loss of generality. We first need to derive the incentive constraints to exert effort on each project. At date 1, a GP who worked at date 0 exerts effort if

$$\begin{aligned} & p^2 w^{SS}(x) + p(1-p) [w^{SF}(x) + w^{FS}(x)] \\ & \geq pqw^{SS} + p(1-q)w^{SF}(x) + (1-p)qw^{FS}(x) + B(1-x) \end{aligned} \quad (30)$$

which we can express as $F_1 \geq B(1-x)$ where F_1 is implicitly defined by (30). It will also be useful to write the off-equilibrium payoff at date 1 of a GP who shirked at date 0. This payoff is given by

$$\tilde{V}_1(w(x), x) = \max \left\{ pqw^{SS}(x) + q(1-p)w^{SF}(x) + (1-q)pw^{FS}(x), \right. \\ \left. q^2w^{SS} + q(1-q)[w^{SF}(x) + w^{FS}(x)] + B(1-x) \right\} \quad (31)$$

Finally, the incentive constraint at date 0 writes

$$p^2w^{SS}(x) + p(1-p)[w^{SF}(x) + w^{FS}(x)] \geq \tilde{V}_1(w(x), x) + Bx \quad (32)$$

which we rewrite as $F_0 \geq Bx$ where F_0 is implicitly defined by (32).

Suppose then that the right inequality of (27) does not bind and $w^{SF}(x) > 0$. We can then decrease $w^{SF}(x)$ without affecting any constraint while keeping the GP expected payoff constant. Suppose first that $w^{FS}(x) = 0$ and consider the following change in the compensation schedule

$$\Delta w^{SF}(x) = -p\epsilon, \quad \Delta w^{SS}(x) = (1-p)\epsilon$$

Since $w^{FS}(x) = 0$ and $w^{SF}(x) > 0$ by assumption, the monotonicity constraint (28) is still satisfied for $\epsilon > 0$ small enough. Let us then show that (30) and (32) still hold. For this, it is enough to verify that the changes ΔF_0 and ΔF_1 are positive. We have

$$\Delta F_1 = p(p-q)(\Delta w^{SS}(x) - \Delta w^{SF}(x)) = p(p-q)\epsilon > 0 \\ \Delta F_0 \geq (p-q)(p\Delta w^{SS}(x) - (1-p)\Delta w^{SF}(x)) \\ = (p-q)(p(1-p)\epsilon - p(1-p)\epsilon) = 0$$

Finally, we need to check that the GP expected payoff is the same. We have

$$\Delta W = p^2\Delta w^{SS}(x) + p(1-p)\Delta w^{SF}(x) = p^2(1-p)\epsilon - p^2(1-p)\epsilon = 0$$

which proves the claim when $w^{SF}(x) = 0$.

To finish the proof, we need to consider the case when $w^{FS}(x) > 0$. Similar steps show that the following change in the compensation schedule

$$\Delta w^{SF}(x) = \Delta w^{FS}(x) = -p\epsilon, \quad \Delta w^{SS}(x) = 2(1-p)\epsilon$$

for $\epsilon > 0$ small enough. also leaves the GP's expected payoff unchanged while still satisfying the incentive constraints (30) and (32). We will now show that the expected compensation is minimized for $x = \frac{1}{2}$.

A.1.2 Optmality of split investment

We now prove that it is always (weakly) optimal to split investment equally across projects. To prove this result, we distinguish two cases based on the parameter ranges in Lemma 2.

Case 1) $R \geq \frac{2B}{p^2 - q^2}$.

We guess and verify that the right inequality of monotonicity constraint (27) does not bind. Step 1 shows that we can set $w_F^{FS}(x) = w^{SF}(x) = 0$ without loss of generality. The incentive constraint at date 1, equation (30) becomes

$$w^{SS} \geq \frac{B(1-x)}{p(p-q)} \quad (33)$$

The incentive constraint at date 0, equation (32), becomes

$$w^{SS} \geq \begin{cases} \frac{B}{p^2 - q^2} & \text{if } x \in \left[\frac{1}{2}, \frac{p}{p+q}\right] \\ \frac{Bx}{p(p-q)} & \text{if } x > \frac{p}{p+q} \end{cases} \quad (34)$$

Given that $x \geq \frac{1}{2}$, it is clear that the incentive constraint at date 0, equation (34) is binding rather than inequality (33). This proves the claim in the main text for this case. It follows that the expected compensation is minimized at $x = \frac{1}{2}$. In fact, one can find that any value $x \in \left[\frac{q}{p+q}, \frac{p}{p+q}\right]$ minimizes the expected compensation which is equal to

$$w^{SS,*} = \frac{B}{p^2 - q^2}$$

Our initial claim that the monotonicity constraint does not bind is verified since $\frac{R}{2} \leq w^{SS,*}$. This corresponds to the second case in Lemma 2.

Case 2) $R < \frac{2B}{p^2 - q^2}$

Observe that in what follows, it must be that the right inequality of monotonicity constraint (27) binds that is

$$w^{SS}(x) = w^{SF}(x) + R(1-x) \quad (35)$$

Using equation (35), we can rewrite the incentive constraint at date 1, equation (30) as

$$(p-q) \{pR(1-x) + (1-p)w^{FS}(x)\} \geq B(1-x) \quad (36)$$

Proof that $\tilde{V}_1(w(x), x)$ is equal to its second argument

To rewrite the incentive constraint at date 0, equation (32), it is useful to show that

that $\tilde{V}_1(w(x), x)$ is equal to its second argument, that is a GP who shirked at date 0 also shirks at date 1. Suppose it is not the case. Then, we have

$$(p - q) \left[qR(1 - x) + (1 - q)w^{FS}(x) \right] > B(1 - x) \quad (37)$$

Observe first that (37) implies that the incentive constraint at date 1, equation (36) holds. Given the upper bound on R , equation (37) may hold only if $w^{FS}(x) > 0$. But then, $w^{FS}(x)$ should be decreased until either (37) is violated or the right hand side of monotonicity constraint (28), that is

$$w^{SF}(x) - w^{FS}(x) \leq R(2x - 1) \quad (38)$$

holds as an equality. Indeed, when (37) holds, decreasing $w^{FS}(x)$ relaxes the incentive constraint at date 0, equation (32).

To prove that $\tilde{V}_1(w(x), x)$ is equal to its second argument, we are left to show that when (38) holds as an equality, then (37) is violated. Suppose it is not the case. We can rewrite the incentive constraint at date 0, equation (32), as

$$(p - q) \left[pR(1 - x) + w^{SF}(x) - pw^{FS}(x) \right] \geq Bx$$

Using equation (38), it becomes

$$w^{FS}(x) \geq \frac{1}{1 - p} \left[\frac{Bx}{p - q} - R(2x - 1) - pR(1 - x) \right] \quad (39)$$

Inequality (39) is implied by (37) but since (39) is the relevant constraint, it means that $w^{FS}(x)$ should be decreased until (37) is violated, which proves our claim. Using the result that $\tilde{V}_1(w(x), x)$ is equal to its second argument, the incentive constraint at date 0 can be rewritten

$$(p - q) \left[(p + q)R(1 - x) + w^{SF}(x) + (1 - p - q)w^{FS}(x) \right] \geq B \quad (40)$$

Proof that $x = 1/2$ is the optimal split

We can now show our result that the minimum expected compensation is attained in $x = \frac{1}{2}$. The relevant incentive constraints are equation (40) at date 0 and (36) at date 1 as well as the monotonicity constraints (28) and (29). Observe first that either (36) binds or (38) binds or $w^{FS}(x) = 0$. Indeed, otherwise, the following change in the compensation schedule.

$$\Delta w_F^{SF}(x) = (1 - p - q)\epsilon, \quad \Delta w_{LT}^{FS}(x) = -\epsilon$$

leaves (40) unchanged and increases the expected compensation by

$$\Delta W = p(1 - p - q)\epsilon - p(1 - p)\epsilon = -pq\epsilon < 0$$

Case 2)-i) Date 1 incentive constraint binds

Suppose first that (36) binds. Since the expected compensation W is an increasing function of $w^{SF}(x)$, the incentive constraint at date 0, equation (40) should bind, that is

$$w^{SF}(x) = \frac{B}{p - q} - (p + q)R(1 - x) - (1 - p - q)w^{FS}(x)$$

or $w^{SF}(x) = w^{FS}(x)$. If $w^{SF}(x) = w^{FS}(x)$, the incentive constraint at date 0 becomes

$$w^{SF}(x) \geq \frac{1}{2 - p - q} \left[\frac{B}{p - q} - (p + q)R(1 - x) \right]$$

where the right hand side is increasing in x . Hence, in order to minimize W , it is optimal to choose $x = 1/2$. If the first statement is true, then constraint (38) imposes that

$$w^{FS}(x) \geq \frac{1}{2 - p - q} \left[\frac{B}{p - q} - (p + q)R(1 - x) - R(2x - 1) \right] \quad (41)$$

where the right hand side is decreasing in x . Since to minimize W , $w^{SF}(x)$ should be as small as possible, equation (42) and thus (38) should hold as an equality. The analysis of the case where (38) holds as an equality shows again that it is not possible to improve over the split $x = 1/2$.

Case 2)-ii) $w^{FS}(x) = 0$

Suppose now that $w^{FS}(x) = 0$ but that (36) is slack. Then $w^{SF}(x)$ is pinned down using (40) as an equality, that is

$$w^{SF}(x) = \frac{B}{p - q} - (p + q)R(1 - x)$$

and must satisfy equation (28), that is $w^{SF}(x) \leq R(2x - 1)$ given that $w^{FS}(x) = 0$. These equations are compatible only if $x \geq \underline{x} > \frac{1}{2}$ where

$$\frac{B}{p - q} - (p + q)R(1 - \underline{x}) = R(2\underline{x} - 1)$$

In the range $[\underline{x}, 1]$, it is optimal to set $x = \underline{x}$ since the expected compensation

$$W = p^2R(1 - x) + pw^{SF}(x) = \frac{pB}{p - q} - pqR(1 - x)$$

is increasing in x . The expected compensation at $x = \underline{x}$ is given by

$$W = \frac{pB}{p-q} - \frac{qp}{2-p-q} \left[R - \frac{B}{p-q} \right]$$

This is the same as the expected compensation in Lemma 1 and thus does not improve on the split with $x = 1/2$

Case 2)-iii) Monotonicity constraint (38) binds

Suppose finally that $w^{FS}(x) \geq 0$ and that the incentive constraint at date 1, equation (36), is slack but that (38) holds as an equality. Then, the incentive constraint at date 0, equation (40) can be rewritten as

$$w^{FS}(x) \geq \frac{1}{2-p-q} \left[\frac{B}{p-q} - (p+q)R(1-x) - R(2x-1) \right] \quad (42)$$

The condition that $w^{FS}(x) \geq 0$ is equivalent to $x \leq \underline{x}$. Using that the incentive constraint (42) should bind, the expected compensation is equal to

$$W(x) = p^2R(1-x) + p(2x-1)R + p(2-p)w^{FS}(x)$$

Using equation (42), it is easy to see that W is constant over $[1/2, \underline{x}]$. This shows that the expected compensation is minimized for any value $x \in [1/2, \underline{x}]$ and in particular for $x = \frac{1}{2}$ when, the compensation schedule is given by

$$w^{SF} = w^{FS} = \frac{1}{2-p-q} \left[\frac{B}{p-q} - (p+q)\frac{R}{2} \right], \quad w^{SS} = \frac{R}{2} + w^{FS}$$

This finishes the analysis of all possible cases and proves Lemma 2 for $R \leq \frac{2B}{p^2-q^2}$.

A.2 Proof of Corollary 1

– $\Pi_{GP}^*(\lambda)$ and $I^*(\lambda)$ are decreasing in λ

Observe first that $I^*(\lambda)$ is a decreasing function of λ since λ only enters negatively at the denominator. The result about $\Pi_{GP}^*(\lambda)$ immediately follows since Π_{GP} is increasing in I and λ only affects Π_{GP} through I .

– Π_{GP}^* and $I^*(\lambda)$ are increasing in R

The result for $I^*(\lambda)$ in the case $R \geq \frac{2B}{p^2-q^2}$ is immediate since then R only appears negatively at the denominator of $I^*(\lambda)$ in expression (13). The result for Π_{GP}^* again

follows from the observation that R only affects Π_{GP}^* through $I^*(\lambda)$. When $R < \frac{2B}{p^2-q^2}$, let us rewrite the fund size as

$$I^*(\lambda) = \frac{1}{1 - (1 - \lambda)[pR - W^*(R)]} = \frac{1}{1 - (1 - \lambda)\frac{p(2-p)}{2-p-q} \left[R - \frac{B}{p-q} \right]} \quad (43)$$

where we used the expression for $W^*(R)$ in equation (9). Observe that $I^*(\lambda)$ is also increasing in R in this case. The derivative is in fact higher since and increase in R relaxes the monotonicity constraint. Finally, we can write the expected profit of the GP as

$$\Pi_{GP}^*(\lambda) = \frac{W^*(R)}{1 - (1 - \lambda)[pR - W^*(R)]}$$

Observe that *i*) $\Pi_{GP}^*(\lambda)$ is increasing in R and decreasing in $W^*(R)$ and *ii*) $W^*(R)$ is decreasing in R . This proves that the total derivative of $\Pi_{GP}^*(\lambda)$ with respect to R is strictly positive.

– Π_{GP}^* and $I^*(\lambda)$ are increasing in q when $p = q + \alpha$ with α constant

Let us focus first on the case when $R \geq \frac{2B}{p^2-q^2}$. Then we can write the fund size as

$$I^*(\lambda) = \frac{1}{1 - (1 - \lambda)(q + \alpha) \left[R - \frac{B(q+\alpha)}{\alpha(2q+\alpha)} \right]}$$

Since

$$\frac{\partial(q + \alpha)}{\partial q} > 0, \quad \frac{\partial \left(\frac{q+\alpha}{2q+\alpha} \right)}{\partial q} < 0$$

the fund size $I^*(\lambda)$ is increasing in q . Using expression (43) and substituting $p = q + \alpha$, a similar result obtains when $R < \frac{2B}{p^2-q^2}$.

Let us now write the GPs' profit as a function of q , we have

$$\Pi_{GP}^*(q) = \frac{W(q)}{1 - (1 - \lambda)[(q + \alpha)R - W(q)]}$$

so that Π_{GP}^* is increasing in q if and only if

$$W'(q) + (1 - \lambda)R(W(q) - (q + \alpha)W'(q)) \geq 0$$

Let us consider first the case where $R \geq \frac{2B}{p^2-q^2}$. Since then $W'(q) > 0$, it is enough to show that $W(q) - (q + \alpha)W'(q) \geq 0$. We have indeed

$$\begin{aligned} W(q) - (q + \alpha)W'(q) &= \frac{(q + \alpha)^2}{\alpha(2q + \alpha)}B - (q + \alpha)\frac{2q}{\alpha(2q + \alpha)^2}B \\ &= \frac{(q + \alpha)^2}{\alpha(2q + \alpha)^2}(2q + \alpha - 2q) \geq 0 \end{aligned}$$

A.3 Proof of Proposition 2

We first argue that all GPs offer the same contract to LPs. All GPs must offer the compensation schedule derived in Lemma 2, that minimizes their expected compensation W per unit of investment. Otherwise, since Π_{GP} is decreasing in W , a decrease in W would increase the size of the fund and the GP's total profit. Without loss of generality, we can also assume that GPs choose the split $x = 1/2$ between projects.

We now determine the equilibrium fund size I^* . Observe that λ -LPs only invest if their net expected return $r_{LP}(\lambda)$ is greater or equal than zero. Proposition 1 shows that this is the case if the fund size is smaller or equal than $I^*(\lambda)$. Since $I^*(\lambda_L) > I^*(\lambda_H)$, the maximum fund size is $I^*(\lambda_L)$ which is the investment capacity of all GPs when facing only λ_L -LPs, that is when $m_L = M$.

When $m_L > I^*(\lambda_L)$, GPs can reach their maximum investment capacity and the market clears at $I^* = I^*(\lambda_L)$. When $m_L \in [I^*(\lambda_H), I^*(\lambda_L)]$, investment is constrained by the resources available to λ_L -LPs. Hence, the maximum fund size, given that λ_H -LPs do not invest is m_L . Observe that since $m_L \geq I^*(\lambda_H)$, it is indeed optimal for λ_H -LPs not to invest since $r_{LP}(\lambda_H) < 0$. This proves that $I^* = m_L$.

Consider finally the case when $m_L < I^*(\lambda_H)$. Then, λ_H -LPs must invest. Indeed, otherwise, GPs can raise funds of size at most m_L by catering to λ_L -LPS. But since $m_L < I^*(\lambda_H)$, both types of LPs would earn a strictly positive profit and λ_H -LPS would invest, increasing the fund size over m_L . This means that the equilibrium fund size must be $I^* \geq I^*(\lambda_H)$. It cannot be strictly above $I^*(\lambda_H)$ since otherwise $r_{LP}(\lambda_H) < 0$ and λ_H -LPs would not invest. The analysis of these three cases justifies equation (16) for the fund size.

Finally, we need to determine the net expected return $r_{LP}^*(\lambda_L)$ for λ_L -LPs. From equation (15), we have

$$\begin{aligned} r_{LP}^*(\lambda_L) &= \frac{(1-\lambda)(pR-W)I^*}{I^*-1} - 1 = \frac{(I^*(\lambda_L)-1)I^*}{I^*(\lambda_L)(I^*-1)} - 1 \\ &= \frac{I^*(\lambda_L) - I^*}{I^*(\lambda_L)(I^*-1)} \end{aligned}$$

where from the first to the second equality, we used the definition of $I^*(\lambda_L)$ from Proposition 1. This justifies the expression for $r_{LP}^*(\lambda_L)$ in equation (17) and concludes the proof.

A.4 Proof of Proposition 3

The first step is to prove that the market for secondaries clears at price $P_f = 1$ when GPs raise capital from LPs. Using equation (24) and substituting for \bar{P}_f using equation

(18), we obtain

$$\begin{aligned} & \lambda(I-1)P_f + (1-\lambda)(pR-W)I = (1-\lambda)(pR-W)IP_f + \lambda(I-1)P_f^2 \\ \Leftrightarrow & \left[\lambda(I-1)P_f + (1-\lambda)(pR-W)I \right] (1-P_f) = 0 \end{aligned}$$

Since the term between brackets is positive, it must be that $P_f = 1$.

We now derive the expression for $\underline{\lambda}_{SM}$, assuming first that secondaries trade at par, that is $P_f = \bar{P}_f = 1$. Using equation (24), we obtain

$$I_{SM}^*(\lambda) = \frac{1}{1 - (pR - W)} = I^*(0)$$

We need to verify the conjecture that secondaries trade at par. Condition (21) is equivalent to

$$\begin{aligned} (1-\lambda)(M - (I_{SM}^*(\lambda) - 1)) & \geq \lambda(I_{SM}^*(\lambda) - 1) \\ \Leftrightarrow \lambda & \leq 1 - \frac{I_{SM}^*(\lambda) - 1}{M} \\ \Leftrightarrow \lambda & \leq 1 - \frac{(pR - W)}{M[1 - (pR - W)]} =: \underline{\lambda}_{SM} \end{aligned}$$

Hence, when $\lambda \leq \underline{\lambda}_{SM}$, the fund size is given by $I_{SM}^*(\lambda) = I^*(0)$ and secondaries trade at par.

Let us now consider the case $\lambda \geq \underline{\lambda}_{SM}$. Two outcomes are possible. First, GPs may not raise any capital from LPs. In that case, their profit is $\Pi = pR$. We now derive the equilibrium when GPs raise capital from LPs to characterize their optimal choice. The analysis of the previous paragraphs shows that $P_f = 1$ and that there must be cash-in-the-market pricing when $\lambda \geq \underline{\lambda}_{SM}$. Then, using equation (22), we obtain the fund size

$$I^*(\lambda) = 1 + (1 - \lambda)M$$

The discount for secondaries is given by

$$\frac{P_f}{\bar{P}_f} = \frac{I_{SM}^*(\lambda) - 1}{(pR - W)I_{SM}^*(\lambda)} = \frac{(1 - \lambda)M}{(pR - W)(1 + (1 - \lambda)M)}$$

Finally, in order to derive $\bar{\lambda}_{SM}$, we need to check the condition that GPs prefer to raise capital from LPs rather than investing on their own. This holds if

$$\begin{aligned} & WI^*(\lambda) \geq pR \\ \Leftrightarrow & (1 - \lambda)MW \geq pR - W \\ \Leftrightarrow & \lambda \leq 1 - \frac{pR - W}{MW} =: \bar{\lambda}_{SM} \end{aligned}$$

References

- Albuquerque, Rui A., Johan Cassel, Ludovic Phalipou, and Enrique J. Schroth, 2017, Liquidity provision in the secondary market for private equity fund stakes.
- Axelsson, Ulf, Per Strömberg, and Michael S. Weisbach, 2009, Why are buyouts levered? the financial structure of private equity funds, *The Journal of Finance* 64, 1549–1582.
- Diamond, Douglas W., 1984, Financial intermediation and delegated monitoring, *The Review of Economic Studies* 51, 393–414.
- Diamond, Douglas W, and Philip H Dybvig, 1983, Bank Runs, Deposit Insurance, and Liquidity, *Journal of Political Economy* 91, 401–419.
- Harris, Robert S., Tim Jenkinson, and Steven N. Kaplan, 2014, Private equity performance: What do we know?, *The Journal of Finance* 69, 1851–1882.
- Hochberg, Yael V., Alexander Ljungqvist, and Annette Vissing-Jørgensen, 2014, Informational holdup and performance persistence in venture capital, *The Review of Financial Studies* 27, 102–152.
- Holmstrom, Bengt, and Jean Tirole, 1997, Financial intermediation, loanable funds, and the real sector, *The Quarterly Journal of Economics* 112, 663–691.
- Kaplan, Steven N., and Antoinette Schoar, 2005, Private equity performance: Returns, persistence, and capital flows, *The Journal of Finance* 60, 1791–1823.
- Korteweg, Arthur, and Morten Sorensen, 2017, Skill and luck in private equity performance, *Journal of Financial Economics* 124, 535 – 562.
- Lerner, Josh, and Antoinette Schoar, 2004, The illiquidity puzzle: theory and evidence from private equity, *Journal of Financial Economics* 72, 3 – 40.
- Sorensen, Morten, Neng Wang, and Jinqiang Yang, 2014, Valuing private equity, *The Review of Financial Studies* 27, 1977–2021.