Discount Window Stigma and the Term Auction Facility

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Abstract

During the subprime mortgage crisis, banks avoided borrowing from the Fed’s discount window (DW) for fear of a stigma associated with borrowing from the Fed, but actively participated in its Term Auction Facility (TAF) and willingly paid interest rates higher than the rate readily available in the DW. This paper provides theoretical, empirical, and policy analyses of lending of last resort in the presence of borrowing stigmas. First, we provide a model with endogenous stigmas associated with borrowing from different facilities to explain how the combination of DW and TAF increased banks’ borrowings and willingnesses to pay for loans from the Fed. Second, empirical analysis of the detailed data on DW borrowing and TAF bidding from 2007 to 2010 released after the crisis support the theoretical predictions of the model. Finally, we discuss the design of lending-of-last-resort policies in the presence of endogenous stigmas.

Keywords: discount window stigma, auction, adverse selection, lending of last resort

JEL: G01, D44, E58
Access to adequate short-term funding is vital for the stability of individual banks and the entire financial system. The discount window (DW) has been the primary facility for the Fed to offer short-term loans to depository institutions since its inception in 1913. However, it was under-used when the interbank market froze in late 2007, the onset of the financial crisis.

A stigma is believed to be associated with DW borrowing. Banks fear that, once detected of having borrowed from the DW, they would be perceived as financially distressed, a disastrous signal to depositors, creditors, counterparties, and regulators. ¹ As suggestive evidence, banks have regularly paid more for loans on the interbank market than they could readily get through the DW (Peristiani, 1998; Furfine, 2001; Armantier et al., 2015).

In response to the credit crunch and banks’ reluctance to borrow from the DW, the Fed created a temporary program, the Term Auction Facility (TAF), in December 2007. The TAF held an auction every other week, providing a pre-announced amount of collateralized loans, with its participants subject to identical eligibility criteria as well as identical loan maturity and collateral margins as those in DW borrowing. Figure 1a shows the outstanding balance of TAF borrowing versus DW borrowing during the financial crisis. Clearly, the TAF provided much more liquidity than the DW. More puzzling is the fact that banks were willing to pay a higher interest rate to obtain liquidity through the auction. Figure 1b shows that the stop-out rate – the rate that cleared the auction – was higher than the concurrent discount rate – the rate readily available in the DW – in 21 out of the 60 auctions from 2007 to 2010, especially from March to September 2008.

This episode raises a series of questions about the Fed’s lending-of-last-resort policies. Where did the stigma associated with DW borrowing come from? Shouldn’t a similar stigma also prevent banks from participating in the TAF, but why could the TAF overcome the stigma and generate more borrowing than the DW? How did banks decide to borrow from the DW and/or the TAF, and was there any systematic difference between the banks who borrowed from the DW and those who

¹Although the Fed does not disclose publicly which institutions have received loans from the DW, the Board of Governors publishes weekly the total amount of DW lending by each of the twelve Federal Reserve Districts. Therefore, a surge in total DW borrowing could send the market scrambling to identify the loan recipients. Because of the interconnectedness of the interbank lending market, it is not impossible for other banks to infer which institutions went to the Discount Window. Market participants and social media can also infer from other activities.
borrowed from the TAF? Could the system be improved to more effectively provide liquidity to banks in need? The answers to these questions remain unclear, even to policy makers involved with the design of the TAF (Armantier and Sporn, 2013; Bernanke, 2015).

This paper provides a comprehensive analysis of liquidity lending of last resort in the presence of borrowing stigmas (during a financial crisis as well as during normal times) from theoretical, empirical, and policy perspectives.

Theoretically, our framework (i) explicitly models the dynamic decisions of banks participating in the DW and/or in the TAF, and (ii) includes an endogenous stigma cost associated with each facility. The stigma cost is determined by the financial strengths of the banks borrowing from each facility; the worse the participating banks are, the higher the cost is. In contrast, in most of the previous studies, there is no explicit stigma cost in any facility (Philippon and Skreta, 2012; La’O, 2014), or there is no separate stigma cost for different programs because auction is not explicitly modeled (Ennis and Weinberg, 2013; Ennis, 2017; Che et al., 2018).²

Specifically, we introduce a two-week model in which banks of heterogeneous financial conditions may borrow liquidity from the DW or the TAF. The TAF is held in the beginning of the second week and the DW is available throughout the two weeks; that is, before, during, and after

²One notable exception is Gauthier et al. (2015), but the paper has a static model and does not consider the possibility that a bank can participate in the DW after the TAF.
the TAF. Banks are heterogeneous in their financial conditions: weaker banks have more urgent need for liquidity and thus enjoy higher benefits from borrowing. Banks’ financial conditions – their types – are private information. Stronger banks are reluctant to borrow from the DW, because, if detected, they could be perceived as weaker ones. Banks who borrow through the DW, if any, are among the weakest ones in the economy.

We show that the introduction of the auction can mitigate the stigma and expand the set of potential borrowing banks. Banks who participate in the auction submit a bid, an interest rate they are willing to pay for one unit of liquidity. The Fed then collects all bids, ranks them, and allocates funds from the highest bid to the lowest one. Banks who receive funding from the TAF are subject to the same risk of being caught borrowing and subsequently suffer a penalty, but the market has separate beliefs about banks who borrow through the two programs.

Banks endogenously separate themselves and self select into different programs: (1) the weakest banks borrow from the DW immediately, (2) moderately weak banks bid in the TAF and borrow from the DW if they lose in TAF, (3) moderately strong banks bid in the TAF and do not borrow from the DW if they lose in TAF, and (4) the strongest banks do not borrow at all. The clearing price of the auction is uncertain: it can be higher or lower than the discount rate. If many banks turn out to be moderately weak so that they bid in the auction, the stop-out rate of TAF can be higher than the discount rate.

Empirically verifying the theoretical predictions of the model is difficult, because banks’ financial strengths are private information. We use data on borrowing from the DW and bidding and collateral in the TAF released after the crisis to verify (i) that the banks borrowing from the DW are financially worse than the banks borrowing from the TAF, and (ii) that the theoretical prediction that banks borrowing from the DW pay a higher stigma cost than those borrowing from the TAF, who pay a higher stigma cost than those not borrowing. First, using several proxies for financial strengths, we show that among banks who bid in the TAF, losers were on average stronger than winners; banks who bid higher in the TAF were also stronger than those who bid lower (the average CDS spreads of losers were only half of the average spreads of winners). Second, we
compare the banks who borrowed from the DW within three days before a TAF auction with the banks who borrowed from the DW during other times. Since the TAF schedule was announced weeks before the auction date, tapping the discount window right before the auction was a strong signal for the a bank being financially weak. Indeed, using an event-study approach, we confirm that such borrowings were associated with more negative abnormal returns ex-post.

Finally, we present comparative statics and discuss the optimal dynamic design of lending-of-last-resort policies. The introduction of the TAF certainly expands the set of the banks who try to obtain liquidity and the set of the banks who may obtain liquidity, thus potentially increasing the supply of short-term credit to the economy.

In summary, the paper is related to and contributes to the theoretical, empirical, and policy analyses of lending of last resort.

First, our paper contributes theoretically to the literature on discount window stigma. Peristiani (1998), Furfine (2001), and Furfine (2003) offer evidence that banks prefer the Federal Funds Market to the DW, suggesting the existence of the DW stigma. More recently, Armantier et al. (2015) show that more than half of the TAF participants submitted bids above the discount rate during the 2007-2008 financial crisis. They further quantify the cost of the stigma: banks were willing to pay a premium of 44 basis points to avoid borrowing from the DW. Ennis and Weinberg (2013) introduce a model which incorporates search frictions in the interbank lending. The stigma arises endogenously because banks who borrow from the discount window could be either those who were unable to find a counterparty due to search friction, or those who were rejected a loan contract by informed counterparty due to their credit risk.

Second, there are several empirical studies of the TAF. However, where they focus on the real effects of liquidity provision, our paper focuses on what banks obtain the liquidity. McAndrews et al. (2017) and Wu (2011) study the effect of the TAF and conclude that it was effective in lowering Libor and reducing liquidity concern in the interbank lending market. Moore (2017) find that the TAF had a benefit on the real economy: those marginal winners in the auction extended on average 8% more loan growth than marginal losers. Gauthier et al. (2015) construct a model in which strong
banks choose to access the TAF as a way to signal their quality. In their model, TAF borrowing is more costly in the sense that it is less flexible. In contrast, in our model, the TAF is more costly only for bad banks who are desperate for liquidity. Good banks find TAF borrowing less costly for two reasons. First, the stop-out rate is on average lower than the discount rate. Second, the stigma associated with the TAF is lower than the stigma associated with the DW.

Finally, our paper contributes to the policy analysis of the government’s intervention in market plagued by adverse selection and improves the understanding of interventions during the financial crisis, which better prepares ourselves if the next one arrives. The key difference from previous studies (Philippon and Skreta, 2012; Tirole, 2012; Gale and Yorulmazer, 2013; Fuchs and Skrzypacz, 2015) is the consideration of an endogenous stigma.\(^3\)

Subsequently, we present our theoretical analysis in section 1, empirical analysis in section 2, and policy analysis in section 3.

1 Theoretical Analysis

1.1 Environment

There are \(n\) risk-neutral banks in the economy. Each bank \(i\) has a type \(\theta_i \in [0, 1]\), which represents a bank’s financial strength. Throughout the paper, we assume \(\theta_i\) is private information only known by the bank itself (we drop subscript \(i\) whenever no confusion arises). Each bank’s type \(\theta\) is identically and independently distributed according to a continuous and strictly increasing distribution function \(F\), with density function \(f\). We describe the most general functions of benefit and stigma cost of borrowing, and provide specific microfoundations for these general functions. Our microfoundations are able to capture the essence of various existing models in the literature.

1.1.1 Benefit of Borrowing

A type-\(\theta\) bank who borrows one unit of liquidity with (gross) interest rate \(r\) receives a benefit \(b(\theta, r)\). Assume the benefit is strictly decreasing in type and in interest rate. Assume the benefit is

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\(^3\)Recent work by Che et al. (2018) considers the government’s optimal bailout policy in the presence of an endogenous stigma in a dynamic setting, but does not consider the role of auction.
weakly submodular in type and interest rate: the marginal benefit of a lower interest rate is higher for weaker banks. We present two microfoundations of the borrowing benefit function.

**Microfoundation 1: Guard Against Adverse Shock (Holmström and Tirole, 1998).** Each bank is endowed with a unit of illiquid assets that will not mature until a later period. The asset generates cash flows $R$ upon maturity but nothing if liquidated early. Before maturity, each type-$\theta$ bank may be hit with an idiosyncratic liquidity shock with probability $1 - \theta$. Before the realization of the liquidity shock, a bank may borrow to guard against the adverse shock. If a bank does not borrow, its expected return is $\theta R$. If a bank can secure a loan at interest rate $r$, then its return is $R - r$. Therefore, the net benefit of borrowing is $b(\theta, r) = (1 - \theta) R - r$. Clearly, the net borrowing benefit is lower when a bank is stronger and/or if the interest rate is higher. The benefit function is separable (hence, weakly submodular) in $\theta$ and $r$.\(^4\)

**Microfoundation 2: No Skin in the Game (Philippon and Skreta, 2012; Tirole, 2012).** The private type $\theta$ represents each bank’s probability of being able to pay back a loan. Whether or not a bank can pay back a loan is not ex-ante verifiable, but is ex-post verifiable and ex-ante contractible. For a bank, the benefit of being able to pay back is smaller than the benefit of not being able to pay back. Let the benefit of paying back be $B$ and let the additional cost of paying back be $\varepsilon > 0$. Therefore, the expected benefit of taking out a loan for a type-$\theta$ bank is $B - \theta \varepsilon$. The expected repayment to a loan with (gross) interest rate $r$ is $\theta r$. The expected net benefit of a loan with interest rate $r$ is $b(\theta, r) = B - \theta (\varepsilon + r)$. Clearly, the benefit function is strictly decreasing in $\theta$ and $r$, and is strictly submodular in $\theta$ and $r$.

1.1.2 Stigma Cost of Borrowing

A bank believed to have a type distributed according to a cumulative distribution function $G$ incurs a stigma cost $k(G)$ from borrowing. Suppose the stigma cost is higher for worse banks:

\(^4\)Furthermore, departing from Holmström and Tirole (1998), we can also incorporate the possibility that even if a bank has borrowed, it only defrays the shock with some exogenous probability $q \in (0, 1)$ (this can be due to mismatch between liquidity demand and supply, poor internal governance of the bank, or late delivery of liquidity). In this case, the expected payoff of not borrowing is $\theta R$. A bank can pay back only when the return is realized, so the expected payoff of borrowing is $[\theta + (1 - \theta)q](R - r)$. Therefore, the net benefit is $b(\theta, r) = (1 - \theta)q R - (1 + \theta)q r$, which is strictly decreasing in $\theta$ when $r \leq R$ and strictly decreasing in $\theta$, as well as strictly submodular in $\theta$ and $r$.\}
\( k(G) > k(G') \) if \( G \) is strictly first-order stochastically dominated by \( G' \). We present two microfoundations of the stigma cost function.

**Microfoundation 1: Decline in Stock Price.** A bank may face a decline in stock price due to the low confidence by the market, predatory trading (La’O, 2014), or a higher probability of facing a bank run. One example of a stigma cost function is a constant minus the expected type:

\[
k(G) = k - \int \theta dG(\theta) \equiv k - E_G.
\]

**Microfoundation 2: Higher Interest Rate in the Interbank Market (Gauthier et al., 2015; Ennis, 2017; Che et al., 2018).** Suppose the bank borrows in the competitive interbank market in the next period. A competitive lender who offers a loan with (gross) interest rate \( r \) to type-\( \theta \) bank expects to recover both the principal and the interest with probability \( \theta \) and to incur a loss of 1 with probability \( 1 - \theta \). The expected profit from lending to a bank believed to have a type distributed according to \( G \) is \( rE_G - 1 \). Since the lending market is competitive, the competitive interest rate offered to a type-\( G \) bank is \( r(G) = 1/E_G \). If the borrowing benefit of a type-\( \theta \) bank taking out a loan with interest rate \( r \) is \( b(\theta, r) = (1 - \theta)R - r \), then the expected borrowing benefit of a type-\( \theta \) bank believed to have a type distributed according to \( G \) is \( (1 - \theta)R - r(G) \). The stigma cost associated with being believed to be of a type distributed according to \( G \) is effectively \( k(G) = r(G) = 1/E_G \), inversely related to the posterior mean type. Clearly, \( k(G) > k(G') \) if \( G \) is strictly first-order stochastically dominated by \( G' \), because \( E_G < E_{G'} \). This microfoundation is also empirically grounded by the fact that the interbank rates offered to banks participating in different borrowing facilities were different: Gauthier et al. (2015) show that the banks who borrowed from the TAF in 2008 paid 31 basis points less than the banks who borrowed from the DW in 2008.

**1.2 Discount Window Only**

Discount window borrowing is always available. It offers a loan with (gross) interest rate \( r_D \), which we call discount rate. A bank who borrows from discount window (eventually) pays \( r_D \) and receives a benefit \( b(\theta) \). In addition, a bank may be detected of borrowing and may pay a...
discount window detection penalty $k_D$. In contrast, a bank may detected of not borrowing and may pay a non-borrowing penalty $k_N$. These penalties depend on the market perception about the bank. Let $G_D$ denote the distribution of types of banks borrowing from the discount window. Then $k_D = k(G_D)$. Similarly, $k_N = k(G_N)$. Let $k : \Delta[0, 1] \to \mathbb{R}$ satisfy that worse-perceived banks, in the first-order stochastic dominance sense, receive a bigger penalty, i.e. $k(G) > k(H)$ if $G$ is first-order stochastically dominated by $H$.

Given $k_D$ and $k_N$, a bank borrows if and only if

$$b(\theta) - r_D - k_D \geq -k_N,$$

that is

$$b(\theta) \geq r_D + k_D - k_N.$$

Regardless of the detection penalties, worse banks have higher incentives to borrow. Therefore, there is a cutoff $\theta_1$ such that banks worse than $\theta_1$ prefer borrowing from the discount window to borrowing from the auction.

Given that banks $[0, \theta_1]$ borrow from DW and banks $(\theta_1, 1]$ do not borrow,

$$G_D(\theta) = \frac{F(\theta)}{F(\theta_1)} \quad \forall \theta \leq \theta_1$$

$$G_N(\theta) = \frac{F(\theta) - F(\theta_1)}{1 - F(\theta_1)} \quad \forall \theta > \theta_1$$

Clearly, $G_D$ is first-order stochastically dominated by $G_N$, so $k_D > k_N$: there is an endogenous stigma associated with discount window borrowing. Given the threshold borrowing strategies, $k_D$ and $k_N$ are both characterized by a single cutoff $\theta_1$.

In equilibrium, $\theta_1$ is determined by

$$b(\theta_1) - r_D - [k_D(\theta_1) - k_N(\theta_1)] = 0.$$
(To guarantee interior solution, assume \( b(0) - r_D - [k(0) - k(1)] > 0 \) and \( b(1) - r_D - [k(1) - k(0)] < 0 \).) There is a unique equilibrium if the left hand side is downward sloping for all \( \theta_1 \), that is, economically, the decline in benefit by type is faster than the decline in stigma by type.

**Theorem 1.** There exists a unique perfect Bayesian equilibrium in which banks \([0, \theta_1] \) borrow from the discount window and banks \((\theta_1, 1] \) do not borrow, when the following monotonicity holds:

\[
  b'(\theta_1) - (k_{D1} - k_{N1}) < 0. 
\]

(Mon-DW)

Furthermore, \( \theta_1 \in (0, 1) \) if

\[
  b(0) - r_D - [k(0) - k(1)] > 0, \quad \text{(Boundary-0)}
\]

\[
  b(1) - r_D - [k(1) - k(0)] < 0. \quad \text{(Boundary-1)}
\]

The total expected borrowing is \( nF(\theta_1) \).

**Proposition 1 (Comparative Statics).** Lowering the discount rate \( r_D \) and/or lowering the detection penalty difference \( k_D(\theta) - k_N(\theta) \) increases borrowing from the discount window.

### 1.3 Discount Window and Term Auction Facility

TAF ran sixty auctions every two weeks between December 2007 and March 2010. We model the coexistence of DW and TAF programs as follows. DW is available at the beginning of period 1. If a bank borrows from DW, it gets exactly the same payoff as in the discount-window-only case: benefit \( b(\theta) \), interest payment \( r_D \), and detection penalty \( k_D \). If a bank does not borrow from DW right away, it can choose to participate in the subsequent auction. The Fed sets up an auction that awards \( m \) units of liquidity and has a minimum bid of \( r_A \). In the auction, each bank submits a bid \( \beta \) above \( r_A \), specifying an interest rate. In reality, Fed aggregates all the bids and awards up to \( m \) winners, and the interest rate of each loan is the lowest bid among the winners, essentially a first-price auction. Following Revenue Equivalence Theorem, we model the auction without loss as a second-price auction in which the interest rate is the highest bid among the losers if there is
any loser, and is the minimum required bid if there is no loser. If a bank does not win in the auction, it can still borrow from the subsequent DW. Since there are a few days of delay by participating in the auction, there is a delay cost associated with borrowing. The benefit of borrowing becomes $\delta \cdot b(\theta)$.

We solve the optimal action by backward induction. A type-$\theta$ bank borrows in the DW after TAF if

$$\delta b(\theta) - r_D - k_D > 0$$

Let $\theta_2$ denote the type of the bank that is indifferent between borrowing and not borrowing.

In the auction, each bank submits its willingness to pay. If a bank’s type is worse than $\theta_2$, then it will borrow in the subsequent DW, so its maximal willingness to pay is determined by

$$\delta b(\theta) - \beta(\theta) - k_A = \delta b(\theta) - r_D - k_D.$$ 

That is,

$$\beta(\theta) = r_D + k_D - k_A \quad \forall \theta \leq \theta_2.$$ 

Note that a bank would bid higher than the discount rate $r_D$ if $k_D > k_A$, that is, if banks borrowing from discount window are perceived worse than banks borrowing from auction. We will show that this will hold in equilibrium. For any bank better than $\theta_2$, its willingness to pay is determined by

$$\delta b(\theta) - \beta(\theta) - k_A = 0.$$ 

That is,

$$\beta(\theta) = \delta b(\theta) - k_A \quad \forall \theta > \theta_2.$$ 

For banks not sufficiently better than $\theta_2$, they will also bid above the discount rate if the shadow cost of discount window is higher than that of the auction, $k_D > k_A$. Assume $r_A$ is sufficiently low
so that \( r_D + k_D - k_A > r_A \). A bank participates in the auction if \( \beta(\theta) \geq r_A \), that is,

\[
\delta b(\theta) - k_A - r_A \geq 0.
\]

Let \( \theta_A \) denote the marginal auction participant.

Now we consider the key participation decision between DW and TAF. Let \( H(\tau) \) represent the cumulative distribution of the highest losing type. For a type-\( \theta \) bank, the bank’s payoff from DW is

\[
b(\theta) - r_D - k_D.
\]

The (continuation) payoff from TAF is

\[
u_A(\theta|H) = \left\{ \begin{array}{ll}
\delta b(\theta) - \int_0^{\theta_2} [\beta(\tau) + k_A] dH(\tau) - \int_{\theta_2}^{\theta_A} [\beta(\tau) + k_A] dH(\tau) - \int_{\theta_A}^{1} [r_A + k_A] dH(\tau) & \theta \leq \theta_2, \\
\int_{\theta}^{\theta_2} [\delta b(\theta) - \beta(\tau) - k_A] dH(\tau) + \int_{\theta_A}^{1} [\delta b(\theta) - r_A - k_A] dH(\tau) & \theta > \theta_2.
\end{array} \right.
\]

The two expressions are respectively simplified to

\[
u_A(\theta|H) = \left\{ \begin{array}{ll}
\delta b(\theta) - k_A - \int_0^{1} \max\{\beta(\tau), r_A\} dH(\tau) & \theta \leq \theta_2, \\
(1 - H(\theta))[\delta b(\theta) - k_A] - \int_0^{1} \max\{\beta(\tau), r_A\} dH(\tau) & \theta > \theta_2.
\end{array} \right.
\]

The slopes of these expressions are \( \delta b'(\theta) \) and \( (1 - H(\theta))\delta b'(\theta) \), respectively. The delaying cost, \( (1 - \delta)b(\theta) \), is always higher for worse banks. Therefore, worse banks strictly prefer to borrow from DW. Let \( \theta_D \) denote the type of the bank that is indifferent between discount window and auction given \( H, \theta_A, k_D, \) and \( k_A \).

In summary, optimal borrowing decisions are characterized by three cutoffs \( \theta_2, \theta_A, \) and \( \theta_D \). In any equilibrium, banks \([0, \theta_D]\) borrow from DW1, banks \([\theta_D, \theta_A]\) participate in the auction, and among the losing banks, banks worse than \( \theta_2 \) borrow from DW2; it is indeterminate if \( \theta_2 > \theta_D \). Banks \([0, \theta_D]\) and possibly some banks \([\theta_D, \theta_2]\) borrow from the discount window, so \( k_D \) is characterized by \( \theta_D \) and \( \theta_2 \). Banks \([\theta_D, \theta_A]\) might win from the auction, and there might be ties.
and rationing among banks \( [\theta_D, \theta_2] \), so \( k_A \) is characterized by \( \theta_D, \theta_2, \) and \( \theta_A \). In particular, \( G_D \) is first-order stochastically dominated by \( G_A \), so for any equilibrium belief, \( k_D > k_A \): the stigma is an endogenously higher against borrowing in the DW than against borrowing from the TAF.

Hence, an equilibrium is described by \( \theta_2, \theta_A, \) and \( \theta_D \).

\[
\delta b(\theta_2) - r_D - k_D(\theta_D, \theta_2) = 0 \quad \text{(DW2)}
\]

\[
\delta b(\theta_A) - r_A - k_A(\theta_D, \theta_2, \theta_A) = 0 \quad \text{(TAF)}
\]

\[
b(\theta_D) - r_D - k_D(\theta_D, \theta_2) = \delta b(\theta_D) - k_A(\theta_D, \theta_2, \theta_A) - \int_{\theta_D}^{1} \max \{ \beta(\tau), r_A \} dH(\tau; \theta_D) \quad \text{(DW1)}
\]

where

\[
h(\theta; \theta_D) = (n - 1)f(\theta) \left( \frac{n - 2}{m - 1} \right) [F(\theta) - F(\theta_D)]^{m-1}[1 - F(\theta) + F(\theta_D)]^{n-m-1},
\]

and

\[
H(\tau; \theta_D) = \int_{\theta_D}^{\tau} h(\theta; \theta_D) d\tau \quad \forall \tau \geq \theta_D.
\]

(DW1) can be reformulated as

\[
b(\theta_D) - r_D - k_D(\theta_D, \theta_2) = \delta b(\theta_D) - \delta B(\theta_D, \theta_2, \theta_A) \quad \text{(DW1‘)}
\]

where

\[
B(\theta_D, \theta_2, \theta_A) = \begin{cases} 
\int_{\theta_D}^{\theta_2} b(\theta_2) dH(\tau; \theta_D) + \int_{\theta_2}^{\theta_A} b(\tau) dH(\tau; \theta_D) + \int_{\theta_A}^{1} b(\theta_A) dH(\tau; \theta_D) & \theta_2 > \theta_D \\
\int_{\theta_D}^{\theta_A} b(\theta_A) dH(\tau; \theta_D) + \int_{\theta_A}^{1} b(\theta_A) dH(\tau; \theta_D) & \theta_2 \leq \theta_D
\end{cases}.
\]

There exists a unique equilibrium given similar monotonicity conditions as in the DW-only case.

**Theorem 2.** There exists a unique perfect Bayesian equilibrium in which banks \([0, \theta_D]\) borrow
from DW in the first period, banks \([0, \theta_A]\) bid in the auction, and banks \([0, \theta_2]\) borrow from DW in the second period, if they have not borrowed, when the following monotonicity conditions hold:

\[
\delta b'(\theta_2) - k_{D2}(\theta_D, \theta_2) < 0 \quad \forall \theta_D, \theta_2 \quad \text{(Mon-DW2)}
\]

\[
\delta b'(\theta_A) - k_{AA}(\theta_D, \theta_2, \theta_A) < 0 \quad \forall \theta_D, \theta_2, \theta_A \quad \text{(Mon-TAF)}
\]

\[
\delta b'(\theta_D) - k_{DD}(\theta_D, \theta_2) < 0 \quad \forall \theta_D, \theta_2 \quad \text{(Mon-DW1)}
\]

and

\[
(1 - \delta)b'(\theta_D) + \delta B_D - \frac{\delta b'(\theta_2)(1 - H_2(\theta_2; \theta_D))}{\delta b'(\theta_2) - k_{D2}(\theta_D, \theta_2)} k_{DD}(\theta_D, \theta_2) < 0 \quad \forall \theta_D, \theta_2. \quad \text{(Mon-D)}
\]

Furthermore, \(\theta_D, \theta_2, \theta_A \in (0, 1)\) if \(\delta < 1\),

\[
\delta b(0) - r_D - k(1) > 0, \quad \text{(Boundary-1)}
\]

\[
\delta b(0) - r_A < 0. \quad \text{(Boundary-0)}
\]

2 Empirical Analysis

In this section, we provide empirical evidence consistent with two sets of theoretical predictions of the model. First, we relate banks’ fundamentals to their borrowing behavior, consistent with the predictions of the model. Second, we apply an event-study approach and study how the market reacted to banks’ borrowing decisions, showing the existence of the detection penalty.

Specifically, we classify banks into the following groups. The first group borrowed from the discount window shortly before a TAF auction. Presumably, this group of banks were desperate for liquidity and if detected, the cost of stigma should be the highest. The second group includes banks who borrowed through bidding successfully in TAF. The model predicts them to be relative stronger and thus carried a lower stigma cost. The last group is comprised of banks who tapped
discount window shortly after the auction. According to our model, their liquidity conditions should be further stronger. One caveat though, is that the real world is dynamic, whereas our model only spans two periods. Consequently, banks who borrowed from the discount window shortly after TAF might also be those who were newly hit with liquidity shocks. If we take this effect into account, the comparison between the second and the last group will be ambiguous.

Throughout the empirical exercise, we combine several datasets. The first one is obtained through Bloomberg and includes 407 institutions that borrowed from the Federal Reserve between Aug 1, 2007 and Apr 30, 2010. These data were released by the Fed on Mar 31, 2011, under a court order, after Bloomberg filed a lawsuit against Fed board for information disclosure (Torres, 2011). The institutions are mostly banks (≈ 73%), together with insurance companies, savings and loans, and other financial service firms. Foreign banks who borrowed through their U.S. subsidiaries were also included. The data contain information on each institution’s daily outstanding balance of its borrowing from the discount window, the Term Auction Facility as well as five other related programs. With this dataset, we are able to identify the exact date of borrowing and examine how the market responded. The second dataset provides details on all 60 TAFs, including names of bidders (both winners and losers), their bidding rates, the amount awarded, as well as the amount of different types of collaterals pledged to back these loans. We obtain this data by filing a Freedom of Information Act (FOIA) request to the Federal Reserve.

Table 1 provides summary statistics of the Bloomberg data. Out of the 407 borrowing institutions, 313 of them were banks, with the rest being either diversified financial services (mostly asset management firms), insurance companies and savings and loans. Among them, 92 borrowers were foreign banks who borrowed mainly through their U.S. subsidiaries. Banks’ choices of borrowing facilities were quite heterogeneous. While a majority (260 out of 407) tapped both facilities, some of them only used one. The total borrowing frequencies also exhibit sharp heterogeneity: some banks never tapped the discount window, one bank used it a total of 242 times. On average, TAF offered more liquidity (3174 million) than DW (1529 million), consistent with the evidence shown in figure 1. However, the bank that borrowed the most from DW took out a total of approximately
Table 1: Summary Statistics from Bloomberg

<table>
<thead>
<tr>
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<th>Mean</th>
<th>Max</th>
<th>Min</th>
<th>SD</th>
<th>10th</th>
<th>50th</th>
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<td></td>
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<td>Diversified Financial Services</td>
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<td>No. of Foreign Banks</td>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>only borrow DW</td>
<td>18</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>only borrow TAF</td>
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<td></td>
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<tr>
<td>borrow both</td>
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<td>borrow neither</td>
<td>43</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>DW borrowing frequency</td>
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<td>0</td>
<td>28.7</td>
<td>0</td>
<td>2</td>
<td>35</td>
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<td>DW before TAF frequency</td>
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<td>42</td>
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<td>4.5</td>
<td>0</td>
<td>0</td>
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<td>DW after TAF frequency</td>
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<td>57</td>
<td>0</td>
<td>5.7</td>
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<tr>
<td>TAF borrowing frequency</td>
<td>5</td>
<td>28</td>
<td>0</td>
<td>5.1</td>
<td>0</td>
<td>3</td>
<td>13</td>
<td></td>
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<td>Total DW borrowing amount</td>
<td>1529</td>
<td>190155</td>
<td>0</td>
<td>10393.8</td>
<td>0</td>
<td>20</td>
<td>1809</td>
<td></td>
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<tr>
<td>Total TAF borrowing amount</td>
<td>3174</td>
<td>100167</td>
<td>0</td>
<td>10727.5</td>
<td>0</td>
<td>58</td>
<td>7250</td>
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<tr>
<td>Number of days in debt to Fed</td>
<td>323</td>
<td>814</td>
<td>28</td>
<td>196.8</td>
<td>85</td>
<td>306</td>
<td>606</td>
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</table>

$190 billion loans over the three-year period, exceeding its counterpart in TAF ($\approx$ $100 billion). This evidence also suggests that DW banks were in more need of liquidity than TAF banks.

We merge this dataset with each borrower’s daily stock market returns and study how the market reacted to borrowing decisions. Specifically, we adopt an event-study approach with the following specifications. The estimation window is set as the period before the interbank market froze up: Jan 3, 2005 to Aug 1, 2007. Predicted returns are estimated using both the market model and Fama-French three-factor model. The choice of event window is more subtle. given that all borrowings were presumably anonymous, and it remains unclear when the borrowing event was actually “detected” by the market. To circumvent this issue, we try estimation windows with different lengths. Our benchmark results will be based on the cumulative abnormal returns (CAR) in the week (five trading days) following the borrowing event, based on the assumption that one source of detection is the weekly public report of aggregate DW borrowings.
Table 2: CAR following Borrowing Events

<table>
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<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DW</td>
<td>-0.009***</td>
<td>-0.015**</td>
<td>0.004</td>
<td>-0.005</td>
</tr>
<tr>
<td>(DW/TAF)</td>
<td>(0.002)</td>
<td>(0.008)</td>
<td>(0.007)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>(TAF/DW)</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>TAF</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>N</td>
<td>2948</td>
<td>209</td>
<td>257</td>
<td>720</td>
</tr>
</tbody>
</table>

Table 2 compares the CAR following each borrowing event. Column (1) shows the results across all discount window borrowing events. Following a DW borrowing, a borrower’s stock return experienced a total of 0.9% in the next week. This effect is significant, both economically and statistically. It is consistent with the well-perceived belief that discount window borrowing carries a stigma. Column (2) and (3) further show the results when a bank borrowed from discount window before and after TAF. Column (2) implies the “stigma” cost was even higher if a bank borrowed right before TAF: an average bank’s stock return experienced a 1.5% drop over the next week. By contrast, if a bank tapped DW within the three-day window after TAF, its CAR was not statistically different from zero. The evidence seems to suggest that borrowing from the DW after TAF did not carry as much a stigma. Finally, Column (4) examines whether there was a “stigma” associated with TAF borrowing. While the CARs in the following week were on average negative, we cannot reject the null hypothesis that it was zero.

Next, we turn to the detailed data on TAF auctions. Table 3 offers a summary of the dataset. There are a total of 434 banks who ever submitted their bids. Among them, 22 were classified as Global systemically important (G-SIBs), and 82 were foreign. Indeed, G-SIBs and foreign banks made on average more bids than the rest of the sample, suggesting that their liquidity positions could be in bigger troubles (Benmelech, 2012).

A key variable that we will use to proxy banks’ fundamentals is the type of collateral they pledge. Specifically, we classify collateral into two groups by their margin requirements. The first groups includes corporate market instruments, non-agency mortgage-backed securities and asset-backed securities. These assets faced more transparency and were therefore more illiquid.
Table 3: Summary Statistics of TAF

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Mean</th>
<th>Max</th>
<th>Min</th>
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<td></td>
</tr>
<tr>
<td>No. of G-SIBs</td>
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<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>No. of Foreign Banks</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total # of Bid: all</td>
<td>13</td>
<td>95</td>
<td>1</td>
<td>13.9</td>
<td>1</td>
<td>8</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>G-SIBs</td>
<td>27</td>
<td>95</td>
<td>1</td>
<td>24.5</td>
<td>1</td>
<td>25</td>
<td>57</td>
<td></td>
</tr>
<tr>
<td>Foreign Banks</td>
<td>25</td>
<td>95</td>
<td>1</td>
<td>18.5</td>
<td>4</td>
<td>23</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>All: high-haircut</td>
<td>0.19</td>
<td>1.00</td>
<td>0.00</td>
<td>0.3</td>
<td>0.00</td>
<td>0.00</td>
<td>0.79</td>
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<td>G-SIBs: high-haircut</td>
<td>0.35</td>
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<td>0.00</td>
<td>0.22</td>
<td>0.92</td>
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<tr>
<td>Foreign: high-haircut</td>
<td>0.40</td>
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<td>0.00</td>
<td>0.4</td>
<td>0.00</td>
<td>0.34</td>
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<tr>
<td>All: low-rating</td>
<td>0.19</td>
<td>1.00</td>
<td>0.00</td>
<td>0.3</td>
<td>0.00</td>
<td>0.00</td>
<td>0.72</td>
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<tr>
<td>G-SIBs: low-rating</td>
<td>0.30</td>
<td>0.97</td>
<td>0.00</td>
<td>0.3</td>
<td>0.00</td>
<td>0.28</td>
<td>0.73</td>
<td></td>
</tr>
<tr>
<td>Foreign: low-rating</td>
<td>0.33</td>
<td>1.00</td>
<td>0.00</td>
<td>0.3</td>
<td>0.00</td>
<td>0.32</td>
<td>0.83</td>
<td></td>
</tr>
</tbody>
</table>

As a result, they had more haircut when pledged collateral. All other types of collateral belong to the second group. Therefore, borrowers, ceteris paribus, should be more reluctant to pledge them. Table 3 shows that G-SIBs and foreign banks pledged on average more high-haircut collateral, again confirming that they liquidity conditions were more severe. An additional variable that we use is the fraction of pledged securities that were rated below A. The last three rows of Table 3 show that G-SIBs and foreign banks pledged more of these low-rating collaterals than others.

Table 4 shows the correlation between a bank’s bidding rate and the type of collateral it pledged. In each auction, we classify all bidding banks into two groups: those who bid above and below the median rate that was submitted. The idea behind this test is to use the type of collateral as a proxy for a borrower’s liquidity position. Column (1), (3), and (5) use the whole sample, whereas (2), (4), and (6) focus on the subsample in which the total submitted bids exceeded the amount of liquidity provided. We control for fixed-effects of G-SIBs, foreign banks, as well as each auction. Results are significant both economically and significantly. Consistent with the theoretical prediction, banks who bid above the median rate pledged on average 3 - 15% more collateral that had high haircut. The results are robust if we replace the classification with the actual bidding
rates submitted. We obtain similar results if we replace the dependent variable as the fraction of low-rating securities pledged.

Finally, we study whether TAF participation decisions were correlated. The idea behind this test is to use a bank’s future participation to proxy its current liquidity positions. If a bank had successfully won in the auction, its liquidity should be reduced. Compared to banks who lost, the relative incentive to borrow should decrease, at least before the newly-issued TAF loan matured. However, Tables 5 and 6 show that winners were more likely to bid again in the next two auctions and if they bid, they were more likely to bid above the median rate submitted. Consistent with these results, Table 7 shows among winners, banks who bid higher were also more likely to bid again in the next two auctions. This piece of evidence suggests that 1) compared to losing banks, winners’ liquidity needs were much higher; and 2) banks who bid higher in the auction have more severe liquidity than those who bid lower.
### Table 5: Auction Winner and Future Bidding

<table>
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<th>(4)</th>
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<tbody>
<tr>
<td>Winner</td>
<td>0.032**</td>
<td>0.075***</td>
<td>0.078***</td>
<td>0.078***</td>
<td>0.059***</td>
<td>0.065***</td>
</tr>
<tr>
<td></td>
<td>(0.015)</td>
<td>(0.017)</td>
<td>(0.019)</td>
<td>(0.018)</td>
<td>(0.019)</td>
<td>(0.019)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.822***</td>
<td>0.822***</td>
<td>0.722***</td>
<td>0.722***</td>
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<td>0.698***</td>
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<td>(0.014)</td>
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<td>(0.036)</td>
<td>(0.035)</td>
<td>(0.036)</td>
<td>(0.035)</td>
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<tr>
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<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
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<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
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</tr>
<tr>
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<td>Full</td>
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<td>0.012</td>
<td>0.085</td>
<td>0.049</td>
<td>0.095</td>
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### Table 6: Probability of Bidding High in the Next Two Auctions

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<td>High-rate bidders</td>
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<td>0.412***</td>
<td>0.579***</td>
<td>0.450***</td>
<td>0.543***</td>
<td>0.451***</td>
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<tr>
<td></td>
<td>(0.013)</td>
<td>(0.030)</td>
<td>(0.014)</td>
<td>(0.033)</td>
<td>(0.014)</td>
<td>(0.033)</td>
</tr>
<tr>
<td>Constant</td>
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<td>959</td>
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<td>$R^2$</td>
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Table 7: High-rate Bidder and Future Bidding

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<th>(4)</th>
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<td>High-rate bidders</td>
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<td>(0.011)</td>
<td>(0.027)</td>
<td>(0.012)</td>
<td>(0.027)</td>
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<td>0.793***</td>
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<td>(0.008)</td>
<td>(0.023)</td>
<td>(0.050)</td>
<td>(0.052)</td>
<td>(0.050)</td>
<td>(0.053)</td>
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<td>0.001</td>
<td>0.100</td>
<td>0.035</td>
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<td>0.039</td>
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3 Policy Analysis

We present comparative statics in this section. Suppose that banks $[0, \theta_D]$ borrow from the discount window, and banks $(\theta_D, \theta_A)$ bid in the term auction facility. In this equilibrium, $\theta_D$ and $\theta_A$ are determined by

$$b(\theta_A) - k_A(\theta_D, \theta_A) - r_A - \delta = 0 \quad (A)$$

$$B(\theta_D, \theta_A) - k_D(\theta_D) - r_D = 0 \quad (D)$$

where $B(\theta_D, \theta_A)$ is the expected benefit of the loser with the highest bid when banks $(\theta_D, \theta_A)$ bid in the auction:

$$B(\theta_D, \theta_A) = \int_{\theta_D}^{\theta_A} b(\tau) h(\tau; \theta_D) d\tau + [1 - H(\theta_A; \theta_D)] b(\theta_A),$$

where

$$h(\tau; \theta_D) = (n - 1)f(\tau) \left(\frac{n - 2}{m - 1}\right) [F(\tau) - F(\theta_D)]^{m-1}[1 - F(\tau) + F(\theta_D)]^{n-m-1},$$

$$H(\theta_A; \theta_D) = \int_{\theta_D}^{\theta_A} h(\tau; \theta_D) d\tau.$$

Suppose (A2) holds throughout subsequent comparative statics analyses. In addition, assume

$$b' - k_{AA} < 0 \quad (A1)$$

for any $\theta_D$ and any $\theta_A$. Given assumptions (A1) and (A2), we have

$$(b' - k_{AA})(B_D - k_{DD}) + B_A k_{AD} > 0.$$
collinear equations

\[ A_1 x + B_1 y + C_1 = 0 \]
\[ A_2 x + B_2 y + C_2 = 0 \]

we get

\[ x = \frac{-B_1 C_2 - B_2 C_1}{B_1 A_2 - B_2 A_1}, \quad y = \frac{-A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1}. \]

### 3.1 More TAF Funds

By IFT,

\[(b' - k_{AA})d_A - k_{AD}d_D - k_{Am} = 0 \quad \text{(Am)}\]
\[B_A d_A + (B_D - k_{DD})d_D + B_m = 0 \quad \text{(Dm)}\]

Solving for \(d_D\) and \(d_A\), we get

\[d_D = -\frac{(b' - k_{AA})B_m + k_{Am}B_A}{(b' - k_{AA})(B_D - k_{DD}) + k_{ADB_A}}\]
\[d_A = \frac{k_{Am}(B_D - k_{DD}) - B_m k_{AD}}{(b' - k_{AA})(B_D - k_{DD}) + k_{ADB_A}}\]

Note that \(B_m < 0\): expected payment decreases as more TAF funds are awarded; and \(k_m < 0\): more TAF funds (all else equal) increase the perception about the auction winners and thus decrease the auction detection penalty.

If (A1) and (A2) hold, \(d_D < 0\), that is, more TAF funds lower discount window lending. More TAF funds increase better banks’ probabilities of winning, and thus lead to a lower expected payment of all winning banks. Banks who were originally indifferent between discount window and auction now strictly prefer the auction due to lower expected payment. However, \(d_A\) is ambiguous: more TAF funds do not necessarily increase better banks’ participation in the auction. There are two opposite effects on the auction detection penalty. The first effect: because some banks who originally would borrow from the discount window now participate in the auction, the perception
about banks winning in the auction worsens. The second effect: because more TAF funds are released, banks in better financial position are more likely to win, so the perception about banks winning in the auction improves due to increased TAF funds.

3.2 Less Frequent Auctions

These two variables have exactly the same effects on the equilibrium cutoffs. By IFT,

\[(b' - k_{AA})d_A - k_{AD}d_D - 1 = 0 \quad \text{(ArA)}\]

\[(B_A - k_{DD})d_A + (B_D - k_{DD})d_D = 0 \quad \text{(DrA)}\]

Solving for \(d_D\) and \(d_A\), we get

\[d_D = -\frac{B_A}{(b' - k_{AA})(B_D - k_{DD}) + k_{AD}B_A}\]

\[d_A = -\frac{B_A}{B_D - k_{DD}}d_D\]

As the auction cost \(r_A\) or \(\delta\) increases, more banks choose to borrow from the discount window: \(d_D > 0\), and, if \(B_D - k_{DD} < 0\), some better banks do not participate in the auction, \(d_A < 0\). Because auction cost increases, the expected payment in the auction increases. Banks who were indifferent between the discount window and the auction now prefer the discount window. Because more worse banks go to discount window and do not show up in the auction, the perception for auction winners increases. The auction detection penalty decreases. However, directly, the auction cost increases due to an increase in \(\delta\) or \(r_A\). If the increase in the direct auction cost dominates the reduction in the auction detection penalty, then the banks originally indifferent between auction and not borrowing now strictly prefer not borrowing.

3.3 Lower Discount Rate

By IFT,

\[(b' - k_{AA})d_A - k_{AD}d_D = 0\]
\[(B_D - k_{DD})d_D + B_A d_A - 1 = 0\]

Solving for \(d_D\) and \(d_A\), we get

\[d_D = \frac{b' - k_{AA}}{(b' - k_{AA})(B_D - k_{DD}) + k_{AD}B_A}\]

\[d_A = \frac{k_{AD}}{(b' - k_{AA})(B_D - k_{DD}) + k_{AD}B_A}\]

Both \(d_D\) and \(d_A\) are negative. If the discount rate \(r_D\) increases, fewer banks go to the discount window. Those who skip the discount window participate in the auction and are very likely to win. These banks bring down the perception of the auction winners and thus increase the auction detection penalty. As a result, banks originally indifferent between borrowing and not borrowing no longer want to borrow because of increased auction penalty.

### 3.4 More Transparent Borrowing

Let \(p\) parametrize the detection penalties so that \(k_{Ap} \geq 0\) and \(k_{Dp} \geq 0\). One interpretation is that \(p\) is the probability that a bank is detected of borrowing so that increased detection probability is associated with an increased cost for banks. For example, the implementation of the Dodd-Frank Act can be treated as an increase in auction and discount window detection penalties. By IFT,

\[(b' - k_{AA})d_A - k_{AD}d_D - k_{Ap} = 0\]

\[B_A d_A + (B_D - k_{DD})d_D - k_{Dp} = 0\]

Solving for \(d_D\) and \(d_A\), we get

\[d_D = \frac{(b' - k_{AA})k_{Dp} - B_A k_{Ap}}{(b' - k_{AA})(B_D - k_{DD}) + k_{AD}B_A}\]

\[d_A = \frac{k_{Dp}k_{AD} + (B_D k_{DD})k_{Ap}}{(b' - k_{AA})(B_D - k_{DD}) + k_{AD}B_A}\]
Increased detection penalties for sure lower marginal banks’ incentives to participate in the auction: $d_A < 0$. The change in discount window participation is ambiguous, and depends on the changes in the detection penalties of the two programs. If the penalty for a program does not change while the penalty for the other program increases, the bank originally indifferent between the discount window and auction would choose that program.

Regardless, increased detection penalty, say in the form of public disclosure of borrowing, would never increase aggregate potential participation. The Fed’s resistance to disclosure of discount window borrowing is justified on this ground.
References

Armantier, Olivier and John Sporn, “Auctions Implemented by the Federal Reserve Bank of New York during the Great Recession,” Staff Reports 635, Federal Reserve Bank of New York September 2013.


A Appendix

A.1 Equilibrium Characterization

A.1.1 Discount Window only

Welfare In this case, an individual bank’s expected utility is

\[
U = \int_0^{\theta_1} [(1 - \theta) R - r_D - k_D] f(\theta) d\theta + \int_{\theta_1}^{1} (-k_N) f(\theta) d\theta.
\]

Under uniform distribution,

\[
U = \left(1 - \frac{\theta_1}{2}\right) R\theta_1 - (r_D + k_D)\theta_1 - k_N (1 - \theta_1).
\]

The total welfare of all banks are

\[
W = nU.
\]

A.1.2 TAF only

The equilibrium is characterized by two thresholds \{\theta_2, \theta_A\}:

\[
(1 - \theta_2) R - r_D - k_D - \delta = -k_N \quad (1)
\]

\[
(1 - \theta_A) R - (k_A - k_N) - \delta = r_A. \quad (2)
\]

1. Banks between [0, \theta_A] bid in TAF. Those in [0, \theta_2] make identical bids \(\beta_D(\theta) = r_D + (k_D - k_A)\) and those in [\theta_2, \theta_A] make bids \(\beta_N(\theta) = (1 - \theta) R - (k_A - k_N) - \delta\).

2. Among banks who lose in TAF, those between [0, \theta_2] borrow from discount window.

Let us write the expressions for equilibrium penalties: \(k_D, k_A\) and \(k_N\). Clearly,

\[
k_D = \int_0^{\theta_2} \frac{k(\theta)}{F(\theta_2)} dF(\theta).
\]
With uniform distribution and $k(\theta) = p(1 - \theta)$,

$$k_D = p \left(1 - \frac{\theta_2}{2}\right).$$

Next, we compute $k_A$. In this case, let $\tau$ be the p.d.f for the $m^{th}$ lowest type among all the $n$ banks and $h(\tau)$ be its p.d.f.

$$h(\tau) = \binom{n}{m} \left[\frac{m}{1}\right] \left[F(\tau)\right]^{m-1} \left[1 - F(\tau)\right]^{n-m} f(\tau).$$

If $\tau \in [0, \theta_2]$, then banks who borrow from TAF follow the same distribution as $F(\theta)$ on $[0, \theta_2]$, which leads to the first component of $k_A$:

$$k_{A1} = \int_0^{\theta_2} \mathbb{E}[k(\theta) | \theta \in [0, \theta_2)] h(\tau) d\tau.$$ 

With uniform distribution and $k(\theta) = p(1 - \theta)$,

$$k_{A1} = \int_0^{\theta_2} p \left(1 - \frac{1}{2} \theta_2\right) h(\tau) d\tau.$$ 

If $\tau \in [\theta_2, \theta_A]$, then exactly $m$ banks receive liquidity from TAF, and one of them is $\tau$, with the other $m - 1$ banks follow the same distribution of $F(\theta)$ on $[0, \tau]$. In this case,

$$k_{A2} = \int_{\theta_2}^{\theta_A} \left\{ \frac{1}{m} k(\tau) + \frac{m-1}{m} \mathbb{E}[k(\theta) | \theta \in [0, \tau]] \right\} h(\tau) d\tau.$$ 

With uniform distribution and $k(\theta) = p(1 - \theta)$,

$$k_{A2} = \int_{\theta_2}^{\theta_A} \left\{ \frac{1}{m} p(1 - \tau) + \frac{m-1}{m} p \left(1 - \frac{\tau}{2}\right) \right\} h(\tau) d\tau.$$ 

A2
If $\tau \in [\theta_A, 1]$, then all banks who bid in TAF will receive liquidity. In this case,

$$k_{A3} = \int_{\theta_A}^{1} \mathbb{E}[k(\theta) | \theta \in [0, \theta_A]] h(\tau) d\tau.$$  

With uniform distribution and $k(\theta) = p(1 - \theta)$,

$$k_{A3} = \int_{\theta_A}^{1} p \left( 1 - \frac{1}{2} \theta_A \right) h(\tau) d\tau.$$  

The stigma cost associated with TAF borrowing is therefore

$$k_A = k_{A1} + k_{A2} + k_{A3}.$$  

Now we compute $k_N$. Let us first consider the case that the $m^{th}$ lowest bank is below $\theta_2$. In this case, banks who do not necessarily borrow are those who falls into $[\theta_2, 1]$, which leads to the first component of $k_N$:

$$k_{N1} = \int_{0}^{\theta_2} \mathbb{E}[k(\theta) | \theta \in [\theta_2, 1]] h(\tau) d\tau.$$  

With uniform distribution and $k(\theta) = p(1 - \theta)$,

$$k_{N1} = \int_{0}^{\theta_2} p \left( 1 - \frac{\theta_2 + 1}{2} \right) h(\tau) d\tau.$$  

Next, consider the case when $\tau$ falls into $[\theta_2, \theta_A]$:

$$k_{N2} = \int_{\theta_2}^{\theta_A} \mathbb{E}[k(\theta) | \theta \in [x, 1]] \binom{n}{m} \binom{m}{1} F^{m-1}(x) [1 - F(x)]^{n-m} f(x) dx.$$  

With uniform distribution and $k(\theta) = p(1 - \theta)$,

$$k_{N2} = \int_{\theta_2}^{\theta_A} p \left( 1 - \frac{x + 1}{2} \right) \binom{n}{m} \binom{m}{1} x^{m-1} [1 - x]^{n-m} dx.$$
Finally, when $\tau$ falls into $[\theta_A, 1]$:

$$k_{N3} = [1 - H (\theta_A)] \mathbb{E} [k(\theta) | \theta \in [\theta_A, 1]].$$

6 With uniform distribution and $k(\theta) = p(1 - \theta)$,

$$k_{N3} = p \left(1 - \frac{\theta_A + 1}{2}\right) [1 - H (\theta_A)].$$

The stigma cost associated with non-borrowing is therefore

$$k_N = k_{N1} + k_{N2} + k_{N3}.$$

**Welfare** Let $u(\theta)$ be the an individual expected utility given its type $\theta$. Let

$$g(\tau) = (n-1) \binom{n-2}{m-1} [F(\tau)]^{m-1} [1 - F(\tau)]^{n-m-1} f(\tau).$$

$\tau$ is the $m^{th}$ lowest bank among all the other $n - 1$ banks.

1. If $\theta \in [0, \theta_2]$, then

$$u(\theta) = \int_0^{\theta_2} [(1 - \theta) R - r_D - k_D - \delta] g(\tau) d\tau + \int_{\theta_2}^{\theta} [(1 - \theta) R - (1 - \tau) R - k_N] g(\tau) d\tau$$

$$+ [1 - G(\theta_A)] [(1 - \theta) R - r_A - k_A - \delta].$$

2. If $\theta \in [\theta_2, \theta_A]$, then

$$u(\theta) = \int_0^\theta (-k_N) g(\tau) d\tau + \int_\theta^{\theta_A} [(1 - \theta) R - (1 - \tau) R - k_N] g(\tau) d\tau$$

$$+ [1 - G(\theta_A)] [(1 - \theta) R - r_A - k_A - \delta].$$

6I used to write $k_{N3} = \int_{\theta_A}^1 k(\theta) | \theta \in [\theta_A, 1]] \binom{n}{m} F^{m-1}(x) [1 - F(x)]^{n-m} f(x) dx$ but this is not correct. It fails to take into account that exactly one bank’s type is $\tau$. The results are identical though.
3. If $\theta \in [\theta_A, 1]$, then

$$u(\theta) = -k_N.$$ 

An individual bank’s expected welfare is

$$U = \int_0^1 u(\theta) f(\theta) d\theta.$$ 

Under uniform distribution, this simplifies to

### A.1.3 Both Discount Window and TAF

The equilibrium is characterized by two thresholds $\{\theta_D, \theta_A\}$:

1. Banks between $[0, \theta_D]$ borrow from DW1.

2. Banks between $[\theta_D, \theta_A]$ bid in the auction. They bid exactly $\beta^N(\theta) = (1 - \theta) R - (k_A - k_N) - \delta$.

$$k_A + \delta + \int_{\theta_D}^{\theta_A} \beta(\tau) g(\tau) d\tau + (1 - G(\theta_A)) r_A = r_D + k_D \quad (3)$$

$$\quad (1 - \theta_A) R - (k_A - k_N) - \delta = r_A \quad (4)$$

where $g(\tau) = (n - 1) \binom{n-2}{m-1} (F(x) - F(\theta_D))^{m-1} (1 - F(x) + F(\theta_D))^{n-m-1} f(\tau)$ is the p.d.f. of the highest losing type from the perspective of a type $\theta_D$ bank.

Let us write the expressions for equilibrium penalties: $k_D, k_A$ and $k_N$. Clearly,

$$k_D = \int_{0}^{\theta_D} \frac{k(\theta)}{F(\theta_D)} dF(\theta).$$

With uniform distribution and $k(\theta) = p (1 - \theta)$,

$$k_D = p \left(1 - \frac{\theta_D}{2}\right).$$

Next, we compute $k_A$. In this case, let $\tau$ be the the $m^{th}$ lowest type among all the banks that fall
into \([\theta_D, \theta_A]\) and \(h(\tau)\) be its \(p.d.f.\).

\[
h(\tau) = \binom{n}{m} \binom{m}{1} [F(\tau) - F(\theta_D)]^{m-1} [1 - F(\tau) + F(\theta_D)]^{n-m} f(\tau).
\]

With probability \(h(\tau)\), the penalty shall be \(\mathbb{E} [k(\theta) \mid \theta \in [\theta_D, \tau]]\). Otherwise, the penalty shall be \(\mathbb{E} [k(\theta) \mid \theta \in [\theta_D, \theta_A]]\). Therefore

\[
k_A = \int_{\theta_D}^{\theta_A} \mathbb{E} \left[ \frac{1}{m} k(\tau) + \frac{m-1}{m} k(\theta) \mid \theta \in [\theta_D, \tau] \right] h(\tau) d\tau + \left[ 1 - \int_{\theta_D}^{\theta_A} h(\tau) d\tau \right] \mathbb{E} [k(\theta) \mid \theta \in [\theta_D, \theta_A]].
\]

With uniform distribution and \(k(\theta) = p(1 - \theta)\),

\[
k_A = \int_{\theta_D}^{\theta_A} \left[ \frac{1}{m} p(1 - \tau) + \frac{m-1}{m} p \left( 1 - \frac{1}{2} (\theta_D + \tau) \right) \right] h(\tau) d\tau + \left[ 1 - \int_{\theta_D}^{\theta_A} h(\tau) d\tau \right] p \left( 1 - \frac{1}{2} (\theta_D + \theta_A) \right).
\]

Next, we consider \(k_N\). For \(\tau \in [\theta_D, \theta_A]\), with probability \(h(\tau)\), the penalty shall be \(\mathbb{E} [k(\theta) \mid \theta \in [\tau, 1]]\). Otherwise, the penalty shall be \(\mathbb{E} [k(\theta) \mid \theta \in [\theta_A, 1]]\). Therefore

\[
k_N = \int_{\theta_D}^{\theta_A} \mathbb{E} [k(\theta) \mid \theta \in [\tau, 1]] h(\tau) d\tau + \left[ 1 - \int_{\theta_D}^{\theta_A} h(\tau) d\tau \right] \mathbb{E} [k(\theta) \mid \theta \in [\theta_A, 1]].
\]

With uniform distribution and \(k(\theta) = p(1 - \theta)\),

\[
k_N = \int_{\theta_D}^{\theta_A} p \left( 1 - \frac{1}{2} (1 + \tau) \right) h(\tau) d\tau + \left[ 1 - \int_{\theta_D}^{\theta_A} h(\tau) d\tau \right] p \left( 1 - \frac{1}{2} (\theta_A + 1) \right).
\]

Welfare  

Let

\[
g(\tau) = (n - 1) \binom{n-2}{m-1} [F(\tau) - F(\theta_D)]^{m-1} [1 - F(\tau) + F(\theta_D)]^{n-m-1} f(\tau).
\]

1. If \(\theta \in [0, \theta_D]\), then

\[
u(\theta) = (1 - \theta) R - r_D - k_D.
\]
2. If $\theta \in [\theta_D, \theta_A]$, then
\[
  u(\theta) = \int_{\theta_D}^{\theta} (-k_N) g(\tau) d\tau + \int_{\theta}^{\theta_A} [(1 - \theta) R - \beta^N(\tau) - k_A - \delta] g(\tau) d\tau + [1 - G(\theta_A)] [(1 - \theta) R - r_A - k_A - \delta] .
\]

3. If $\theta \in [\theta_A, 1]$, then
\[
  u(\theta) = -k_N.
\]

An individual bank’s expected welfare is
\[
  U = \int_0^1 u(\theta) f(\theta) d\theta.
\]

Next, we compute the aggregate welfare. Let $M$ be the number of banks whose realized types are in $[\theta_D, \theta_A]$. The aggregate welfare if $M \leq m$ differs from that if $M > m$. Clearly, the probability that a total of $M$ banks fall into $[\theta_D, \theta_A]$ is
\[
  P(M, F(\theta_A) - F(\theta_D), n) = \binom{n}{M} [F(\theta_A) - F(\theta_D)]^M [1 - F(\theta_A) + F(\theta_D)]^{n-M}.
\]

- **Subcase 1**: $M \leq m$. The aggregate welfare is
\[
  W_M = (n - M) \left[ \int_{\theta_D}^{\theta} u(\theta) f(\theta) d\theta + \int_{\theta}^{1} u(\theta) f(\theta) d\theta \right] + M \int_{\theta_D}^{\theta_A} [(1 - \theta) R - r_A - k_A - \delta] f(\theta) d\theta \cdot \frac{1 - F(\theta_A) + F(\theta_D)}{F(\theta_A) - F(\theta_D)}.
\]

- **Subcase 2**: $M > m$. Let $\tau$ be the $m + 1^{th}$ lowest bank among the $M$ banks. Its distribution is
\[
  h(\tau) = \binom{M - 1}{m} \left[ \frac{F(\tau) - F(\theta_D)}{F(\theta_A) - F(\theta_D)} \right]^m \left[ 1 - \frac{F(\tau) - F(\theta_D)}{F(\theta_A) - F(\theta_D)} \right]^{M-1-m} \frac{f(\tau)}{F(\theta_A) - F(\theta_D)}.
\]
The aggregate welfare is

\[
W_M = (n-M) \left[ \int_0^{\theta_D} u(\theta) f(\theta) \, d\theta + \int_{\theta_A}^1 u(\theta) f(\theta) \, d\theta \right] + \frac{1}{1 - F(\theta_A) + F(\theta_D)} - k_N \right) \\
+ \frac{\int_{\theta_D}^1 \left[ (1-\theta)R - \beta^N (\tau) - k_A - \delta \right] g(\tau) \, d\tau f(\theta) \, d\theta}{F(\theta_A) - F(\theta_D)} \\
+ (M-m)(-k_N)
\]

The total welfare is

\[
W = \sum_{M=0}^n P(M, F(\theta_A) - F(\theta_D), n) W_M.
\]

A.2 Proofs

A.2.1 Proof of Theorem 1

Proof. Let \( h_m^n(x) \equiv \binom{n}{m} x^m (1-x)^{n-m} \). Define three correspondences:

\[
\phi_1(\theta_1, \theta_2, \theta_A) = \left\{ \theta : u_1(\theta|\theta_1, \theta_2, \theta_A) - \max\{u_A(\theta|\theta_1, \theta_2, \theta_A), u_N(\theta|\theta_1, \theta_2, \theta_A)\} \geq 0 \right\} \cup \{0\},
\]

\[
\phi_2(\theta_1, \theta_2, \theta_A) = \left\{ \theta : u_2(\theta|\theta_1, \theta_2, \theta_A) - u_N(\theta|\theta_1, \theta_2, \theta_A) \geq 0 \right\} \cup \{0\},
\]

and

\[
\phi_A(\theta_1, \theta_2, \theta_A) = \left\{ \theta : u_A(\theta|\theta_1, \theta_2, \theta_A) - u_N(\theta|\theta_1, \theta_2, \theta_A) \geq 0 \right\} \cup \{0\},
\]

where

\[
u_1(\theta|\theta_1, \theta_2, \theta_A) = (1-\theta)R - r_D - k_D(\theta_1, \theta_2, \theta_A),
\]

\[
u_2(\theta|\theta_1, \theta_2, \theta_A) = (1-\theta)R - r_D - k_D(\theta_1, \theta_2, \theta_A) - \delta,
\]
\( u_A(\theta|\theta_1, \theta_2, \theta_A) = \\
\begin{cases} 
(1 - \theta) R - \int_{\theta_1}^{1} [\max(\beta(\tau), r_A) - k_A(\theta_1, \theta_2, \theta_A)] \, dh_m^{n-1} [F(\tau) - F(\theta_1)] - \delta & 0 \leq \theta \leq \theta_1 \\
\int_{0}^{\theta} [(1 - \theta) R - \max(\beta(\tau), r_A) - k_A(\theta_1, \theta_2, \theta_A)] \, dh_m^{n-1} [F(\tau) - F(\theta_1)] - \delta & \\
+ \int_{0}^{\theta} [-k_N(\theta_1, \theta_2, \theta_A)] \, dh_m^{n-1} [F(\tau) - F(\theta_1)] & \theta_1 \leq \theta \leq \theta_A 
\end{cases} 
\)

and

\[ u_N(\theta|\theta_1, \theta_2, \theta_A) = -k_N(\theta_1, \theta_2, \theta_A). \]

Economically, if it is believed that (i) \([0, \theta_1]\) is the set of banks willing to borrow from discount window 1, (ii) \([0, \theta_A]\) is the set of banks willing to bid if it has not borrowed from discount window 1, and (iii) \([0, \theta_2]\) is the set of banks willing to borrow from discount window 2 if it has not borrowed after auction, then optimally, (i) \(\phi_1(\theta_1, \theta_2, \theta_A)\) is the set of banks willing to borrow from discount window 1, (ii) \(\phi_A(\theta_1, \theta_2, \theta_A)\) is the set of banks willing to bid in the auction if it has not borrowed from discount window 1, and (iii) \(\phi_A(\theta_1, \theta_2, \theta_A)\) is the set of banks willing to borrow from discount window 2 if it has not borrowed after auction. We have an equilibrium if the belief is consistent with the optimal action: \([0, \theta_1] = \phi_1(\theta_1, \theta_2, \theta_A), \ [0, \theta_2] = \phi_2(\theta_1, \theta_2, \theta_A), \ [0, \theta_A] = \phi_A(\theta_1, \theta_2, \theta_A); \) or more simply, if \((\theta_1, \theta_2, \theta_A) \in \phi(\theta_1, \theta_2, \theta_A) \equiv (\phi_1(\theta_1, \theta_2, \theta_A), \phi_2(\theta_1, \theta_2, \theta_A), \phi_A(\theta_1, \theta_2, \theta_A))\). Hence, to prove the existence of an equilibrium, it suffices to show that the correspondence \(\phi \equiv (\phi_1, \phi_2, \phi_A)\) has a fixed point.

Mathematically, each of the three correspondences is well-defined on \(X \equiv [0, 1]^3 \cap \{(\theta_1, \theta_2, \theta_A) : \theta_1 \leq \theta_A\}\), a non-empty, compact, and convex subset of the Euclidean space \(\mathbb{R}^3\), and is upperhemi-continuous with the property that \(\phi_\omega(x)\) for each \(\omega \in \{1, 2, 3\}\) is non-empty, closed, and convex for all \(x \in X\). By Kakutani’s fixed point theorem, \(\phi : X \to 2^X\) has a fixed point \(x \in X\). \(\square\)

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A.2.2 Proof of Theorem 2

Proof. Since $\delta b_2 - k_{D2}$ is negative by (Mon-DW2), given any $\theta_D$, there is a unique $\theta_2$ that satisfies (DW2). Let $\theta_2(\theta_D)$ denote the $\theta_2$ that satisfies (DW2) for any $\theta_D$. We have

$$\delta b_2 \theta_2' - k_{D2} \theta_2' - k_{DD} = 0 \Rightarrow \theta_2'(\theta_D) = \frac{k_{DD}}{\delta b_2 - k_{D2}}.$$

Note that $\theta_2' > 0$ by $k_{DD} < 0$ and (Mon-DW2). Since $\delta b_A - k_{AA} < 0$ by (Mon-TAF), given $\theta_D$ and $\theta_2(\theta_D)$, there is a unique $\theta_A$ that satisfies (TAF). Let $\theta_A(\theta_D)$ denote $\theta_A$ that satisfies (TAF) for any $\theta_D$ and $\theta_2(\theta_D)$. We have

$$\delta b_A \theta_A' - k_{AA} \theta_A' - k_{A2} \theta_A' - k_{AD} = 0 \Rightarrow \theta_A' = \frac{k_{A2} \theta_2' + k_{AD}}{\delta b_A - k_{AA}} = \frac{k_{A2} k_{DD} + k_{AD} (\delta b_2 - k_{D2})}{(\delta b_A - k_{AA}) (\delta b_2 - k_{D2})}.$$

Note that $\theta_A' > 0$ by $k_{A2}, k_{DD}, k_{AD} < 0$, (Mon-TAF) and (Mon-DW2). Finally,

$$(1 - \delta) b(\theta_D) + \delta B(\theta_D, \theta_2, \theta_A) - r_D - k_D(\theta_D, \theta_2)$$

has slope

$$(1 - \delta) b_D + \delta B_D - k_{DD} + \delta B_A \theta_A' + (\delta b_2 - k_{D2}) \theta_2'.$$

We know that

$$B_2 = \begin{cases} b'(\theta_2(\theta_D)) \cdot H(\theta_2(\theta_D); \theta_D) & \theta_2(\theta_D) > \theta_D \equiv b_2 H_2(\theta_2; \theta_D) \\ 0 & \theta_2(\theta_D) \leq \theta_D \end{cases}$$

and

$$B_A = b_A(\theta_A(\theta_D)) \cdot (1 - H(\theta_A(\theta_D); \theta_D)) \equiv b_A(1 - H_A).$$
Plugging in, the slope becomes

\[(1 - \delta)b_D + \delta b_D = \delta b_A(1 - H_A) \frac{k_A 2k_{DD} + k_A d(\delta b_2 - k_{DD})}{(\delta b_A - k_{AA}) (\delta b_2 - k_{DD})} + \frac{\delta b_2 H_2 - k_{DD}}{\delta b_2 - k_{DD}}\]

which simplifies to

\[\Delta \equiv \left[(1 - \delta)b_D + \delta b_D - \frac{\delta b_2 (1 - H_2)}{\delta b_2 - k_{DD}}k_{DD}\right] + \left[\delta b_A(1 - H_A) \frac{k_A 2k_{DD} + k_A d(\delta b_2 - k_{DD})}{(\delta b_A - k_{AA}) (\delta b_2 - k_{DD})}\right].\]

The first term is negative by (Mon-D). The last term is negative by (Mon-DW2) and (Mon-TAF).

If we let \(k_{A2} = 0\), the term becomes

\[\Delta \equiv (1 - \delta)b_D + \delta b_D + \frac{\delta b_A(1 - H_A)}{\delta b_A - k_{AA}}k_{AD} - \frac{\delta b_2 (1 - H_2)}{\delta b_2 - k_{DD}}k_{DD}.\]

\[\square\]

A.2.3 Proof of Proposition 2

Proof. Consider the equilibrium characterized by \(\theta_D, \theta_2, \theta_A\) in (DW2), and (TAF), and (DW1’).

\[\delta b(\theta_2) - r_D - k_D(\theta_D, \theta_2) = 0 \quad \text{(DW2)}\]

\[\delta b(\theta_A) - r_A - k_A(\theta_D, \theta_2, \theta_A) = 0 \quad \text{(TAF)}\]

\[(1 - \delta)b(\theta_D) + \delta B(\theta_D, \theta_2, \theta_A) - r_D - k_D(\theta_D, \theta_2) = 0 \quad \text{(DW1’)}\]

By Implicity Function Theorem,

\[(\delta b_2 - k_{DD})d_2 - k_{DD}d_D - k_{Dm} = 0 \quad \text{(m-DW2)}\]

\[\delta b_A d_A - k_{AAd} d_A - k_{A2} d_2 - k_{ADD} d_D - k_{Am} = 0 \quad \text{(m-TAF)}\]

\[(1 - \delta)b_D d_D + \delta B_D d_D + \delta B_2 d_2 + \delta B_A d_A - k_{DD} d_D - k_{D2} d_2 + \delta B_m - k_{Dm} = 0 \quad \text{(m-DW1)}\]
where $B_m < 0$, $k_{Am} < 0$, and $k_{Dm} > 0$.

$$ (\delta b_2 - k_{D2})d_2 - k_{DD}d_D - k_{Dm} = 0 $$

$$ (\delta b_A - k_{AA})d_A - k_{A2}d_2 - k_{AD}d_D - k_{Am} = 0 $$

$$ ((1 - \delta)b_D + \delta B_D - k_{DD})d_D + (\delta H_2 b_2 - k_{D2})d_2 + \delta(1 - H_A)b_A d_A + (\delta B_m - k_{Dm}) = 0 $$

$$ d_2 = \frac{k_{DD}}{\delta b_2 - k_{D2}}d_D + \frac{k_{Dm}}{\delta b_2 - k_{D2}} $$

$$ d_A = \frac{k_{A2}d_2 + k_{AD}d_D + k_{Am}}{\delta b_A - k_{AA}} = \frac{k_{A2}(k_{DD}d_D + k_{Dm}) + (k_{Am} + k_{AD}d_D)(\delta b_2 - k_{D2})}{(\delta b_A - k_{AA})(\delta b_2 - k_{D2})} $$

(m-DW1) can be written as

$$ ((1 - \delta)b_D + \delta B_D - k_{DD})d_D + (\delta b_2 H_2 - k_{D2})d_2 $$

$$ + \delta B_A \frac{k_{A2}(k_{DD}d_D + k_{Dm}) + k_{Am} + k_{AD}d_D}{(\delta b_A - k_{AA})(\delta b_2 - k_{D2})} + (\delta B_m - k_{Dm}) $$

$$ = \left[ ((1 - \delta)b_D + \delta B_D - k_{DD})d_D + (\delta b_2 H_2 - k_{D2}) \frac{k_{DD}}{\delta b_2 - k_{D2}}d_D \right] + (\delta b_2 H_2 - k_{D2}) \frac{k_{Dm}}{\delta b_2 - k_{D2}} $$

$$ + \delta B_A \frac{k_{A2}k_{DD} + k_{AD}(\delta b_2 - k_{D2})}{(\delta b_A - k_{AA})(\delta b_2 - k_{D2})}d_D + \delta B_A \frac{k_{A2}k_{Dm} + k_{Am}(\delta b_2 - k_{D2})}{(\delta b_A - k_{AA})(\delta b_2 - k_{D2})} + (\delta B_m - k_{Dm}) $$

$$ - \Delta \cdot d_D = \frac{\delta b_A (1 - H_A) k_{A2}}{(\delta b_A - k_{AA})(\delta b_2 - k_{D2})}k_{Dm} - \frac{(\delta b_A - k_{AA})\delta b_2 (1 - H_2)}{(\delta b_A - k_{AA})(\delta b_2 - k_{D2})}k_{Dm} + \frac{\delta b_A (1 - H_A)}{\delta b_A - k_{AA}}k_{Am} + \delta B_m $$

Let $k_{A2} = 0$. 

\( \square \)